

# Some Properties of Strong Solutions to Nonlinear Heat and Moisture Transport in Multi-layer Porous Structures

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## Abstract

The present paper deals with mathematical models of heat and moisture transport in layered building envelopes. The study of such processes generates a system of two doubly nonlinear evolution partial differential equations with appropriate initial and boundary conditions. The existence of the strong solution in two dimensions on a (short) time interval is proven. The proof rests on regularity results for elliptic transmission problem for isotropic composite materials.

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Initial-boundary value problems for second-order parabolic systems, local existence, smoothness and regularity of solutions, coupled heat and mass transport

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## 1. Introduction

Building envelopes, which act as barriers between the indoor and outdoor environments, present a crucial component responsible for the building's performance over the whole service life. In this regard, an important requirement to achieve an energy-efficient design is the assessment of the heat and moisture behavior of the component when exposed to natural climatic conditions.

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This task can hardly be accomplished by purely experimental means, mainly due to the long-term character of environmental variations and the transport processes involved. Therefore, a considerable research effort has been devoted to the development of predictive models for coupled heat and mass transfer in building materials, see e.g. [3, 9] for historical overviews.

The major challenge in predicting the transport phenomena in building components lies in their complex porous microstructure, resulting in an intricate mechanism of moisture absorption from surrounding environment. Here, the dominant physical processes involve adsorption forces, attracting vapor phase molecules to solid parts of the porous system, and capillary condensation in pores. This needs to be complemented with non-linear dependence of thermal conduction on temperature and water content. As a result, engineering models of simultaneous heat and moisture transfer are posed in the form of strongly coupled parabolic system with highly non-linear coefficients. Discretization of these equations, typically based on finite volume or finite element methods, then provides the basis for numerous simulation tools used in engineering practice, see e.g. [11] for a recent survey. However, to our best knowledge, the qualitative properties of the resulting systems remain largely unexplored.

The mathematical models of transport processes in porous composite media consist of the balance equations, governing the conservation of mass (moisture) and thermal energy, supplemented by the appropriate boundary, transmission and initial conditions. This system can be written in the form

$$\frac{\partial B_\ell^j(\mathbf{u}_\ell)}{\partial t} - \nabla \cdot \mathbf{A}_\ell^j(\mathbf{u}_\ell, \nabla \mathbf{u}_\ell) = f_\ell^j(\mathbf{u}_\ell) \quad \text{in } Q_{\ell T}, \quad j = 1, 2, \quad (1)$$

with the nonlinear boundary conditions

$$-\mathbf{A}_\ell^j(\mathbf{u}_\ell, \nabla \mathbf{u}_\ell) \cdot \mathbf{n}_\ell(\mathbf{x}) = g_\ell^j(\mathbf{x}, t, u_\ell^j) \quad \text{on } S_{\ell T}, \quad j = 1, 2, \quad (2)$$

the so-called transmission conditions

$$\begin{cases} u_\ell^j = u_m^j & \text{on } \Sigma_{m\ell}^T, \\ -\mathbf{A}_\ell^j(\mathbf{u}_\ell, \nabla \mathbf{u}_\ell) \cdot \mathbf{n}_\ell(\mathbf{x}) = -\mathbf{A}_m^j(\mathbf{u}_m, \nabla \mathbf{u}_m) \cdot \mathbf{n}_\ell(\mathbf{x}) & \text{on } \Sigma_{m\ell}^T, \end{cases} \quad (3)$$

$j = 1, 2$ , and the initial condition

$$\mathbf{u}_\ell(\mathbf{x}, 0) = \boldsymbol{\mu}_\ell(\mathbf{x}) \quad \text{in } \Omega_\ell. \quad (4)$$

Here,  $\Omega$  represents a two-dimensional bounded domain with a Lipschitz-continuous boundary  $\Gamma = \partial\Omega$ ;  $\mathbf{n} = (n_1, n_2)$  denotes the outer unit normal to  $\Gamma$ .  $\Omega$  consists of  $M$  disjoint subdomains  $\Omega_\ell$  with boundary  $\partial\Omega_\ell$ ,  $\ell = 1, \dots, M$ ,

separated by smooth internal interfaces  $\Gamma_{m\ell} = \partial\Omega_m \cap \partial\Omega_\ell \neq \emptyset$ . For a fixed positive  $T$ , we denote by  $Q_{\ell T}$  the space-time cylinder  $Q_{\ell T} = \Omega_\ell \times (0, T)$ , similarly  $S_{\ell T} = (\partial\Omega_\ell \cap \Gamma) \times (0, T)$  and  $\Sigma_{m\ell}^T = \Gamma_{m\ell} \times (0, T)$ . Further, in (1)–(4),  $\mathbf{u}_\ell = (u_\ell^1, u_\ell^2)$  represents the unknown fields of state variables and the vector  $\boldsymbol{\mu}_\ell = (\mu_\ell^1, \mu_\ell^2)$  describes the initial condition. By  $\mathbf{B}_\ell$ ,  $\mathbf{A}_\ell^j$ ,  $\mathbf{f}_\ell$ ,  $\mathbf{g}_\ell$ , we denote the vectors  $\mathbf{B}_\ell = (B_\ell^1, B_\ell^2)$ ,  $\mathbf{A}_\ell^j = (A_\ell^{j1}, A_\ell^{j2})$ ,  $\mathbf{f}_\ell = (f_\ell^1, f_\ell^2)$ ,  $\mathbf{g}_\ell = (g_\ell^1, g_\ell^2)$ , which are functions of primary unknowns  $\mathbf{u}_\ell$ . Hence, the problem is strongly nonlinear.

The existence of weak solutions to the system (1) in homogeneous bounded domains ( $\ell = 1$ ) subject to mixed boundary conditions with homogeneous Neumann boundary conditions has been shown by Alt and Luckhaus in [2]. They obtained an existence result assuming the operator  $\mathbf{B}$  in the parabolic part to be only (weak) monotone and subgradient. This result has been extended in various different directions. Filo and Kačur [7] proved the local existence of the weak solution for the system with nonlinear Neumann boundary conditions and under more general growth conditions on nonlinearities in  $\mathbf{u}$ . These results are not applicable if  $\mathbf{B}$  does not take the subgradient form, which is typical of coupled heat and mass transport models.

In this context, the only related works we are aware of are due to Vala [26], Li and Sun [20] and Li *et al.* [21]. Nonetheless, even though [26] admits non-symmetry in the parabolic part, it requires unrealistic symmetry in the elliptic term. The latter works, studying a model arising from textile industry, prove the global existence for one-dimensional problem using the Leray-Schauder fixed point theorem. The proofs, however, exploit the specific structure of the model and as such are not applicable to our general setting.

In this paper we adapt ideas presented by Giaquinta and Modica in [8] and Weidemaier in [27], where the local solvability of quasilinear diagonal parabolic systems is proved, to show the local existence of strong solution to the general transmission problem (1)–(4) for isotropic media under less restrictive assumptions on the operator  $\mathbf{B}(\mathbf{u})$  and the parabolicity condition of the problem. The main result (local in time existence) is proved by means of a fixed point argument based on the Banach contraction principle.

The paper is organized as follows. In Section 2, we introduce the appropriate function spaces and recall important embeddings and interpolation-like inequalities needed below together with some auxiliary results. In Section 3, we specify our assumptions on data and structure conditions and introduce the precise definition of admissible domains describing the composite body under which the main result of the paper is proved. In Section 4, we prove the existence and uniqueness of the solution to an auxiliary linearized problem using the regularity result for elliptic systems in composite-like domains. To

make the text more readable, technical details of the proof are collected in Appendices Appendix A, Appendix B and Appendix C. The main result is proved in Section 5 via the Banach contraction principle. Finally, in Section 6 we present applications of the theory to selected engineering models of heat and mass transfer.

## 2. Preliminaries

### 2.1. Definition of some function spaces and notation

We denote by  $\mathbf{W}_\ell^{l,p} \equiv W^{l,p}(\Omega_\ell)^2$ ,  $l \geq 0$  ( $l$  need not to be an integer) and  $1 \leq p \leq \infty$ , the usual Sobolev space of functions defined in  $\Omega_\ell$  and by  $\mathbf{W}_{\ell,\Gamma}^{l-1/p,p} \equiv W^{l-1/p,p}(\partial\Omega_\ell)^2$  the space of traces of functions from  $\mathbf{W}_\ell^{l,p}$  on  $\partial\Omega_\ell$ . We set  $\mathbf{L}_\ell^p \equiv \mathbf{W}_\ell^{0,p}$ . Let  $\mathcal{B}$  be an arbitrary Banach space, then  $(\mathcal{B})^*$  represents its dual.  $\phi'(t)$  indicates the partial derivative with respect to time; we also write

$$\phi'(t) := \frac{\partial \phi}{\partial t}.$$

In order to define the concept of strong solution, we will make use of the following Banach spaces

$$\begin{aligned} \mathcal{X}_{\ell,T} := \{ \phi; \phi'(t) \in L^\infty(0,T; \mathbf{L}_\ell^2), \phi'(t) \in L^2(0,T; \mathbf{W}_\ell^{2,2}), \\ \phi''(t) \in L^2(0,T; \mathbf{L}_\ell^2), \phi(0) = \mathbf{0} \} \end{aligned}$$

and

$$\mathcal{Y}_{\ell,T} := \{ \varphi; \varphi'(t) \in L^2(0,T; \mathbf{L}_\ell^2), \varphi(0) \in \mathbf{W}_\ell^{1,2} \},$$

respectively, equipped with the norms

$$\|\phi\|_{\mathcal{X}_{\ell,T}} := \|\phi'(t)\|_{L^\infty(0,T; \mathbf{L}_\ell^2)} + \|\phi'(t)\|_{L^2(0,T; \mathbf{W}_\ell^{2,2})} + \|\phi''(t)\|_{L^2(0,T; \mathbf{L}_\ell^2)} \quad (5)$$

and

$$\|\varphi\|_{\mathcal{Y}_{\ell,T}} := \|\varphi'(t)\|_{L^2(0,T; \mathbf{L}_\ell^2)} + \|\varphi(0)\|_{\mathbf{W}_\ell^{1,2}}, \quad (6)$$

respectively.

Throughout the paper,  $\ell$  and  $m$  are assumed to always range from 1 to  $M$  and  $m \neq \ell$ , whereas indices  $i, j = 1, 2$ . Unless specified otherwise, we use Einstein's summation convention for indices running from 1 to 2. We shall denote by  $c, c_1, c_2, \dots$  generic constants independent on  $T$  having different values in different places. Let us stress that throughout the paper the function  $C = C(T)$  depends solely on  $T$  and  $C(T) \rightarrow 0_+$  for  $T \rightarrow 0_+$ .

## 2.2. Some embeddings and interpolation like-inequalities

In the paper we shall use the following embeddings (recall that  $\Omega$  is a two-dimensional bounded domain)(see [1, 16]):

$$\begin{cases} \mathbf{W}_\ell^{1,2} \hookrightarrow \mathbf{L}_\ell^p, & \|\phi\|_{\mathbf{L}_\ell^p} \leq c \|\phi\|_{\mathbf{W}_\ell^{1,2}} & \forall \phi \in \mathbf{W}_\ell^{1,2}, 1 \leq p < \infty, \\ \mathbf{W}_\ell^{l,2} \hookrightarrow \mathbf{W}_\ell^{1,p}, & \|\phi\|_{\mathbf{W}_\ell^{1,p}} \leq c \|\phi\|_{\mathbf{W}_\ell^{l,2}} & \forall \phi \in \mathbf{W}_\ell^{l,2}, 1 < l < 2, p = 2/(2-l), \\ \mathbf{W}_\ell^{l,p} \hookrightarrow \mathbf{L}_\ell^\infty, & \|\phi\|_{\mathbf{L}_\ell^\infty} \leq c \|\phi\|_{\mathbf{W}_\ell^{l,p}} & \forall \phi \in \mathbf{W}_\ell^{l,p}, lp > 2. \end{cases} \quad (7)$$

Let us present some properties of  $\mathcal{X}_{\ell,T}$ . Assume  $\phi \in \mathcal{X}_{\ell,T}$ . Using the interpolation inequality [1, Theorem 5.8]

$$\|\phi'(t)\|_{\mathbf{L}_\ell^4} \leq c \|\phi'(t)\|_{\mathbf{W}_\ell^{2,2}}^{1/4} \|\phi'(t)\|_{\mathbf{L}_\ell^2}^{3/4} \quad (8)$$

we obtain

$$\begin{aligned} \|\phi'(t)\|_{L^8(0,T;\mathbf{L}_\ell^4)} &\leq c \|\phi'(t)\|_{L^2(0,T;\mathbf{W}_\ell^{2,2})}^{1/4} \|\phi'(t)\|_{L^\infty(0,T;\mathbf{L}_\ell^2)}^{3/4} \\ &\leq c \|\phi\|_{\mathcal{X}_{\ell,T}}. \end{aligned} \quad (9)$$

For all  $\phi \in \mathcal{X}_{\ell,T}$  we have

$$\begin{aligned} \|\phi\|_{L^\infty(0,T;\mathbf{L}_\ell^\infty)} &\leq c \|\phi\|_{L^\infty(0,T;\mathbf{W}_\ell^{2,2})} \leq c T^{1/2} \|\phi'(t)\|_{L^2(0,T;\mathbf{W}_\ell^{2,2})} \\ &\leq c T^{1/2} \|\phi\|_{\mathcal{X}_{\ell,T}}. \end{aligned} \quad (10)$$

Further, combining (7) and the interpolation inequality [1, Theorem 5.2] we obtain

$$\|\phi'(t)\|_{\mathbf{L}_\ell^\infty} \leq c \|\phi'(t)\|_{\mathbf{W}_\ell^{1,3}} \leq c \|\phi'(t)\|_{\mathbf{W}_\ell^{4/3,2}} \leq c \|\phi'(t)\|_{\mathbf{W}_\ell^{2,2}}^{2/3} \|\phi'(t)\|_{\mathbf{L}_\ell^2}^{1/3} \quad (11)$$

and consequently

$$\|\phi'(t)\|_{L^3(0,T;\mathbf{L}_\ell^\infty)} \leq c \|\phi'(t)\|_{L^3(0,T;\mathbf{W}_\ell^{1,3})} \leq c \|\phi\|_{\mathcal{X}_{\ell,T}}. \quad (12)$$

## 3. Structure conditions and admissible domains

In this Section, we summarize our assumptions on the problem data and specify in detail the geometry of the considered domains.

### 3.1. Structure conditions

- (A1) For every  $\mathbf{z} \in \mathbb{R}^2$ ,  $B_\ell^1(s, z_2)$  and  $B_\ell^2(z_1, s)$  are increasing functions (with respect to  $s$ ),  $B_\ell^j : \mathbb{R}^2 \rightarrow \mathbb{R}$ , such that  $|\partial^\alpha B_\ell^j(\mathbf{z})|$  are bounded on every bounded set in  $\mathbb{R}^2$  for  $|\alpha| \leq 3$ . Further, we denote the matrix

$$b_\ell^{ij}(\mathbf{z}) := \frac{\partial B_\ell^j(\mathbf{z})}{\partial z^i};$$

- (A2)  $\mathbf{A}_\ell^j : \mathbb{R}^2 \times \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^2$  are continuous and of the semilinear form

$$\mathbf{A}_\ell^j(\mathbf{r}, \mathbf{s}) = \sum_{i=1}^2 a_\ell^{ji}(\mathbf{r}) \mathbf{s}_i, \quad (13)$$

for all  $\mathbf{r} \in \mathbb{R}^2$  and  $\mathbf{s}_i = (s_i^1, s_i^2)$ , where  $s_i^j \in \mathbb{R}^2$  for  $i, j = 1, 2$ . Note that in (13)  $\mathbf{r}$  stands for  $\mathbf{u}$  and  $\mathbf{s}_i$  stands for the vector  $\nabla u^i$ . Functions  $a_\ell^{ji} : \mathbb{R}^2 \rightarrow \mathbb{R}$ , are positive, scalar (due to the assumed isotropy of the material) and  $|\partial^\alpha a_\ell^{ji}(\mathbf{r})|$  are bounded on every bounded set in  $\mathbb{R}^2$  for  $|\alpha| \leq 3$ . Further we assume

$$b_\ell^{11}(\boldsymbol{\mu}_\ell) b_\ell^{22}(\boldsymbol{\mu}_\ell) a_\ell^{12}(\boldsymbol{\mu}_\ell) a_\ell^{21}(\boldsymbol{\mu}_\ell) > \left( \frac{b_\ell^{12}(\boldsymbol{\mu}_\ell) a_\ell^{21}(\boldsymbol{\mu}_\ell) + b_\ell^{21}(\boldsymbol{\mu}_\ell) a_\ell^{12}(\boldsymbol{\mu}_\ell)}{2} \right)^2 \quad (14)$$

in  $\overline{\Omega}$  and the ellipticity condition

$$a_\ell^{11}(\boldsymbol{\mu}_\ell) a_\ell^{22}(\boldsymbol{\mu}_\ell) > a_\ell^{12}(\boldsymbol{\mu}_\ell) a_\ell^{21}(\boldsymbol{\mu}_\ell) \quad (15)$$

in  $\overline{\Omega}$  with  $\boldsymbol{\mu}_\ell$  representing the initial distribution of the unknown fields  $\mathbf{u}_\ell$ ;

- (A3)  $\mathbf{f}_\ell : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $|\partial^\alpha f_\ell^j(\mathbf{z})|$  are bounded on every bounded set in  $\mathbb{R}^2$  for  $|\alpha| \leq 2$ ;

- (A4)  $\mathbf{g} : \Gamma \times (0, T) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is of the form of the Newton-type boundary conditions

$$g_\ell^j(\mathbf{x}, t, \mathbf{u}_\ell) = \alpha_\ell^j(u_\ell^j - \sigma^j(\mathbf{x}, t)),$$

where  $\alpha_\ell^j$  are given positive constants and  $\boldsymbol{\sigma} : \Gamma \times (0, T) \rightarrow \mathbb{R}^2$ ,  $j = 1, 2$ ,

$$\boldsymbol{\sigma} \in W^{2,2}(0, T; (\mathbf{W}_\Gamma^{1/2,2})^*) \cap W^{1,2}(0, T; \mathbf{W}_\Gamma^{1/2,2}).$$

### 3.2. Admissible domains

In what follows, we assume that (cf. Figure 1)

- (i)  $\Omega$  is decomposed into nonoverlapping subdomains  $\Omega_\ell$ ;
- (ii) there exists a finite set  $\mathcal{S} \subset \partial\Omega$  of boundary points such that  $\partial\Omega \setminus \mathcal{S}$  is smooth (of class  $C^\infty$ );
- (iii) for every  $P \in \mathcal{S}$  there exists a neighborhood  $\mathcal{U}_P$  and a diffeomorphism  $D_P$  mapping  $\Omega \cap \mathcal{U}_P$  onto  $\mathcal{K}_P \cap B_P$ , where  $\mathcal{K}_P$  is an angle of size  $\omega_P < \pi$  with vertex at the origin (shifted into  $P$ ),

$$\mathcal{K}_P := \{[x_1, x_2] \in \mathbb{R}^2; 0 < r < \infty, 0 < \varphi < \omega_P\},$$

and  $B_P$  is a unit circle centered at the origin ( $r, \varphi$  denote the polar coordinates in the  $(x'_1, x'_2)$ -plane);

- (iv) the interfaces  $\Gamma_{m\ell}$  are smooth (of class  $C^\infty$ ),  $m = 1, \dots, M$ ,  $m \neq \ell$ ;
- (v) there are no cross points of  $\bar{\Gamma}_{m\ell}$  in  $\bar{\Omega}$ .

Let  $\mathcal{M}$  be the set of all boundary points  $A \in \Gamma \equiv \partial\Omega \cap \Gamma_{m\ell}$ ,  $m = 1, \dots, M$ , i.e. the points where any interface  $\Gamma_{m\ell}$  crosses the exterior boundary  $\partial\Omega$ . Further, we assume that

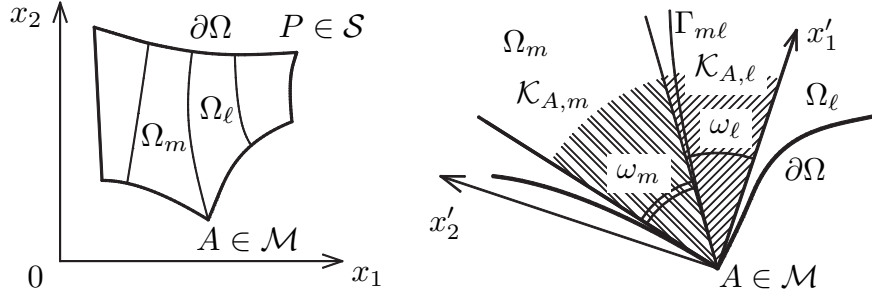


Figure 1: Admissible domains.

- (vi) for every  $A \in \mathcal{M}$ , such that  $A \in \partial\Omega \cap \Gamma_{m\ell}$ , there exists a neighborhood  $\mathcal{U}_A$  and a diffeomorphism  $D_{A,\ell}$  and  $D_{A,m}$ , respectively, mapping  $\Omega_\ell \cap \mathcal{U}_A$  onto  $\mathcal{K}_{A,\ell} \cap B_A$  and  $\Omega_m \cap \mathcal{U}_A$  onto  $\mathcal{K}_{A,m} \cap B_A$ , respectively, where  $\mathcal{K}_{A,\ell}$  and  $\mathcal{K}_{A,m}$ , respectively, is an angle of size  $\omega_\ell$  and  $\omega_m$ , respectively, with vertex at  $A$

$$\mathcal{K}_{A,\ell} := \{[x_1, x_2] \in \mathbb{R}^2; 0 < r < \infty, 0 < \varphi < \omega_\ell\}$$

and

$$\mathcal{K}_{A,m} := \{[x_1, x_2] \in \mathbb{R}^2; 0 < r < \infty, \omega_\ell < \varphi < \omega_\ell + \omega_m\},$$

respectively, and  $B_A$  is a unit circle centered at the origin ( $r, \varphi$  denote the polar coordinates in the  $(x'_1, x'_2)$ -plane with the origin at  $A$ );

(vii) for every  $A \in \mathcal{M}$ , such that  $A \in \partial\Omega \cap \Gamma_{m\ell}$  we have  $\omega_\ell = \omega_m$ ;

(viii) for every  $A \in \mathcal{M}$ ,  $A \in \partial\Omega \cap \Gamma_{m\ell}$ , we have  $\omega_A = \omega_\ell + \omega_m = 2\omega_\ell \leq \pi$ .

*Remark.* Note that conditions (i)–(viii) incorporate, as a special case, rectangular domain composed of regular rectangles  $\Omega_\ell$ . This serves as a basic model for building envelopes, e.g. [11].

#### 4. Solutions to an Auxiliary Linearized System

Following the standard methodology of contraction-based proofs, we consider first an auxiliary linear problem with homogeneous initial condition in the form

$$\beta_\ell^{ji} \frac{\partial u_\ell^i}{\partial t} - \nabla \cdot (\kappa_\ell^{ji} \nabla u_\ell^i) = f_\ell^j(\mathbf{x}, t) \quad \text{in } Q_{\ell T}, \quad (16)$$

$$\kappa_\ell^{ji} \frac{\partial u_\ell^i}{\partial \mathbf{n}_\ell} + \nu_\ell^j u_\ell^j = g_\ell^j(\mathbf{x}, t) \quad \text{on } S_{\ell T}, \quad (17)$$

$$u_\ell^j = u_m^j \quad \text{on } \Gamma_{m\ell} \times (0, T), \quad (18)$$

$$\kappa_\ell^{ji} \frac{\partial u_\ell^i}{\partial \mathbf{n}_\ell} + \kappa_m^{ji} \frac{\partial u_m^i}{\partial \mathbf{n}_m} = 0 \quad \text{on } \Gamma_{m\ell} \times (0, T), \quad (19)$$

$$\mathbf{u}_\ell(\mathbf{x}, 0) = \mathbf{0} \quad \text{in } \Omega_\ell. \quad (20)$$

*Assumptions.* In (16)–(20)  $\nu_\ell^j$  are real positive constants,  $\beta_\ell^{ji} := \beta_\ell^{ji}(\mathbf{x})$ ,  $\kappa_\ell^{ji} := \kappa_\ell^{ji}(\mathbf{x})$  are real positive Lipschitz continuous functions such that

$$\beta_\ell^{11} \beta_\ell^{22} \kappa_\ell^{12} \kappa_\ell^{21} > \left( \frac{\beta_\ell^{12} \kappa_\ell^{21} + \beta_\ell^{21} \kappa_\ell^{12}}{2} \right)^2 \quad \text{in } \overline{\Omega}_\ell \quad (21)$$

and the ellipticity condition

$$\kappa_\ell^{11} \kappa_\ell^{22} > \kappa_\ell^{12} \kappa_\ell^{21} \quad \text{in } \overline{\Omega}_\ell. \quad (22)$$



**Definition 1.** Let  $\mathbf{f}_\ell \in \mathcal{Y}_{\ell,T}$  and  $\mathbf{g}_\ell \in W^{2,2}(0,T;(\mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})^*) \cap W^{1,2}(0,T;\mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})$ . Then  $\mathbf{u}_\ell \in \mathcal{X}_{\ell,T}$  is called a strong solution to the system (16)–(20) iff

$$\begin{aligned} \sum_{\ell=1}^M \int_{\Omega_\ell} \beta_\ell^{ji} \frac{\partial u_\ell^i}{\partial t} v^j \, d\mathbf{x} + \sum_{\ell=1}^M \int_{\Omega_\ell} \kappa_\ell^{ji} \nabla u_\ell^i \cdot \nabla v^j \, d\mathbf{x} + \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \nu_\ell^j u_\ell^j v^j \, dS \\ = \sum_{\ell=1}^M \int_{\Omega_\ell} f_\ell^j v^j \, d\mathbf{x} + \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} g_\ell^j v^j \, dS \end{aligned} \quad (23)$$

holds for every  $\mathbf{v} \in \mathbf{W}^{1,2}$  and almost every  $t \in (0, T)$ .

*Remark.* The regularity in time direction naturally imposes two higher order compatibility conditions on the given functions  $\mathbf{f}_\ell$  and  $\mathbf{g}_\ell$  in (16)–(17). Namely, the first one requires  $\mathbf{g}_\ell(\mathbf{x}, 0)$  to be compatible with (17) while the second one roughly says that  $\mathbf{u}'_\ell(t)|_{t=0}$  has to belong to appropriate Sobolev spaces. This implies additional conditions on  $\mathbf{f}_\ell(\mathbf{x}, 0)$  included in the definition of the space  $\mathcal{Y}_{\ell,T}$ .

**Theorem 4.1.** Let  $\mathbf{f}_\ell \in \mathcal{Y}_{\ell,T}$ ,  $\mathbf{g}_\ell \in W^{2,2}(0,T;(\mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})^*) \cap W^{1,2}(0,T;\mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})$  and  $\mathbf{g}_\ell(\mathbf{x}, 0) = \mathbf{0}$  on  $\partial\Omega$ . Then there exists the unique strong solution  $\mathbf{u}_\ell \in \mathcal{X}_{\ell,T}$  to the system (16)–(20) and the following estimate holds

$$\|\mathbf{u}_\ell\|_{\mathcal{X}_{\ell,T}} \leq c \left( \|\mathbf{f}_\ell\|_{\mathcal{Y}_{\ell,T}} + \|\mathbf{g}_\ell\|_{W^{2,2}(0,T;(\mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})^*)} + \|\mathbf{g}_\ell\|_{W^{1,2}(0,T;\mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})} \right). \quad (24)$$

To prove Theorem 4.1 we prepare the following definitions and lemmas.

**Definition 2** (Problem  $(P_f)$ ). Let us define Problem  $(P_f)$  by the linear transmission system (16)–(20) with  $\mathbf{g}_\ell \equiv \mathbf{0}$  on  $\partial\Omega_\ell \cap \Gamma \times [0, T)$ .

**Lemma 4.2.** Let  $\mathbf{f}_\ell \in \mathcal{Y}_{\ell,T}$ . Then there exists the unique strong solution  $\mathbf{u}_\ell \in \mathcal{X}_{\ell,T}$  of Problem  $(P_f)$ . Moreover, the following estimate holds

$$\|\mathbf{u}_\ell\|_{\mathcal{X}_{\ell,T}} \leq c \|\mathbf{f}_\ell\|_{\mathcal{Y}_{\ell,T}}. \quad (25)$$

*Proof.* See Appendix A. The proof relies on the results for stationary transmission problem presented in Appendix C.  $\square$

**Definition 3** (Problem  $(P_g)$ ). Let us define Problem  $(P_g)$  by the linear transmission system (16)–(20) with  $\mathbf{f}_\ell \equiv \mathbf{0}$  in  $\Omega_\ell \times (0, T)$ .

**Lemma 4.3.** *Let  $\mathbf{g}_\ell \in W^{2,2}(0, T; (\mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})^*) \cap W^{1,2}(0, T; \mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})$  and the compatibility condition  $\mathbf{g}_\ell(\mathbf{x}, 0) = \mathbf{0}$  on  $\partial\Omega$  be satisfied. Then there exists the unique strong solution  $\mathbf{u}_\ell \in \mathcal{X}_{\ell,T}$  of Problem  $(P_g)$  and the following estimate holds*

$$\|\mathbf{u}_\ell(t)\|_{\mathcal{X}_{\ell,T}} \leq c \left( \|\mathbf{g}_\ell\|_{W^{2,2}(0,T;(\mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})^*)} + \|\mathbf{g}_\ell\|_{W^{1,2}(0,T;\mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})} \right). \quad (26)$$

*Proof.* See Appendix B. Similarly as in the proof of Lemma 4.2, we use the results for stationary transmission problem presented in Appendix C.  $\square$

*Proof of Theorem 4.1.* The assertion follows from the superposition principle of the solutions to the linear Problems  $(P_f)$  and  $(P_g)$ .  $\square$

## 5. Solutions to the Nonlinear Parabolic System

**Definition 4** (Problem  $(P_0)$ ). Let us define Problem  $(P_0)$  by the initial-boundary value transmission system (1)–(4) with data and structure conditions satisfying the assumptions (A1)–(A4), see Subsection 3.1.

**Definition 5.** A function  $\mathbf{u}_\ell$ , such that  $\mathbf{u}_\ell'(t) \in L^2(0, T; \mathbf{W}_\ell^{2,2})$  and  $\mathbf{u}_\ell''(t) \in L^2(0, T; \mathbf{L}_\ell^2)$ , is called a strong solution of Problem  $(P_0)$  on  $(0, T)$  with initial data  $\boldsymbol{\mu}_\ell \in \mathbf{W}_\ell^{3,2}$  iff

$$\begin{aligned} \sum_{\ell=1}^M \int_{\Omega_\ell} b_\ell^{ji}(\mathbf{u}_\ell) \frac{\partial u_\ell^i}{\partial t} v^j \, d\mathbf{x} + \sum_{\ell=1}^M \int_{\Omega_\ell} a_\ell^{ji}(\mathbf{u}_\ell) \nabla u_\ell^i \cdot \nabla u_\ell^j \, d\mathbf{x} \\ + \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \alpha_\ell^j(u_\ell^j - \sigma^j) v^j \, dS = \sum_{\ell=1}^M \int_{\Omega_\ell} f_\ell^j(\mathbf{u}_\ell) v^j \, d\mathbf{x} \end{aligned}$$

holds for every  $\mathbf{v} \in \mathbf{W}^{1,2}$  and almost every  $t \in (0, T)$  and

$$\mathbf{u}_\ell(\mathbf{x}, 0) = \boldsymbol{\mu}_\ell(\mathbf{x}) \text{ in } \Omega_\ell.$$

**Theorem 5.1** (Main result). *Let the assumptions (A1)–(A4) be satisfied. For a given  $\boldsymbol{\mu}_\ell \in \mathbf{W}_\ell^{3,2}$ , which is supposed to be compatible with (2)–(3), there exists  $T^* \in (0, T]$  and a function  $\mathbf{u}_\ell$  such that  $\mathbf{u}_\ell$  is the strong solution of Problem  $(P_0)$  on  $(0, T^*)$ .*

Proof of the main result is postponed to the end of this section. We start from a related problem with homogeneous initial condition. To that end, let

$\mathbf{u}_\ell$  be the strong solution of Problem  $(P_0)$  on  $(0, T)$ ,  $\mathbf{u}_\ell = \boldsymbol{\mu}_\ell + \mathbf{y}_\ell$ . Then  $\mathbf{y}_\ell \in \mathcal{X}_{\ell, T}$  and the following equation

$$\begin{aligned} \sum_{\ell=1}^M \int_{\Omega_\ell} b_\ell^{ji}(\boldsymbol{\mu}_\ell + \mathbf{y}_\ell) \frac{\partial y_\ell^i}{\partial t} v^j \, d\mathbf{x} &+ \sum_{\ell=1}^M \int_{\Omega_\ell} a_\ell^{ji}(\boldsymbol{\mu}_\ell + \mathbf{y}_\ell) \nabla(\mu_\ell^i + y_\ell^i) \cdot \nabla v^j \, d\mathbf{x} \\ &+ \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \alpha_\ell^j(\mu_\ell^j + y_\ell^j - \sigma^j) v^j \, dS = \sum_{\ell=1}^M \int_{\Omega_\ell} f_\ell^j(\boldsymbol{\mu}_\ell + \mathbf{y}_\ell) v^j \, d\mathbf{x} \end{aligned}$$

holds for every  $\mathbf{v} \in \mathbf{W}^{1,2}$  and almost every  $t \in (0, T)$ . This amounts to solving the problem with shifted data

$$\begin{aligned} \widehat{b}_\ell^{ji}(\mathbf{x}, \mathbf{y}_\ell) &= b_\ell^{ji}(\mathbf{y}_\ell + \boldsymbol{\mu}_\ell), \\ \widehat{a}_\ell^{ji}(\mathbf{x}, \mathbf{y}_\ell) &= a_\ell^{ji}(\mathbf{y}_\ell + \boldsymbol{\mu}_\ell), \\ \widehat{f}_\ell^j(\mathbf{x}, \mathbf{y}_\ell) &= f_\ell^j(\boldsymbol{\mu}_\ell + \mathbf{y}_\ell). \end{aligned}$$

We often omit the argument “ $\mathbf{x}$ ” writing shortly  $\widehat{a}_\ell^{ji}(\mathbf{y}_\ell)$  instead of  $\widehat{a}_\ell^{ji}(\mathbf{x}, \mathbf{y}_\ell)$ ,  $\widehat{b}_\ell^{ji}(\mathbf{y}_\ell)$  instead of  $\widehat{b}_\ell^{ji}(\mathbf{x}, \mathbf{y}_\ell)$  and  $\widehat{f}_\ell^j(\mathbf{y}_\ell)$  instead of  $\widehat{f}_\ell^j(\mathbf{x}, \mathbf{y}_\ell)$ .

**Definition 6.** Define the operator  $\mathcal{K} : \mathcal{X}_{\ell, T} \rightarrow \mathcal{Y}_{\ell, T}$  given by

$$\begin{aligned} \sum_{\ell=1}^M \int_{\Omega_\ell} \mathcal{K}(\boldsymbol{\phi}_\ell) \cdot \mathbf{v} \, d\mathbf{x} &= \sum_{\ell=1}^M \int_{\Omega_\ell} \left( \widehat{b}_\ell^{ji}(\mathbf{0}) - \widehat{b}_\ell^{ji}(\boldsymbol{\phi}_\ell) \right) \frac{\partial \phi_\ell^i}{\partial t} v^j \, d\mathbf{x} \\ &+ \sum_{\ell=1}^M \int_{\Omega_\ell} \left( \widehat{a}_\ell^{ji}(\mathbf{0}) - \widehat{a}_\ell^{ji}(\boldsymbol{\phi}_\ell) \right) \nabla \phi_\ell^i \cdot \nabla v^j \, d\mathbf{x} \\ &- \sum_{\ell=1}^M \int_{\Omega_\ell} \widehat{a}_\ell^{ji}(\boldsymbol{\phi}_\ell) \nabla \mu_\ell^i \cdot \nabla v^j \, d\mathbf{x} \\ &+ \sum_{\ell=1}^M \int_{\Omega_\ell} \widehat{f}_\ell^j(\boldsymbol{\phi}_\ell) v^j \, d\mathbf{x}, \end{aligned} \tag{27}$$

which holds for every  $\mathbf{v} \in \mathbf{W}^{1,2}$  and almost every  $t \in (0, T)$ .

*Remark.* Let  $\mathbf{u}_\ell = \boldsymbol{\mu}_\ell + \mathbf{y}_\ell$ . The function  $\mathbf{u}_\ell$  is the strong solution of Problem  $(P_0)$  on  $(0, T)$  with initial data  $\boldsymbol{\mu}_\ell \in \mathbf{W}_\ell^{3,2}$  iff for  $\mathbf{y}_\ell \in \mathcal{X}_{\ell, T}$  the following equation

$$\begin{aligned} \sum_{\ell=1}^M \int_{\Omega_\ell} \widehat{b}_\ell^{ji}(\mathbf{0}) \frac{\partial y_\ell^i}{\partial t} v^j \, d\mathbf{x} &+ \sum_{\ell=1}^M \int_{\Omega_\ell} \widehat{a}_\ell^{ji}(\mathbf{0}) \nabla y_\ell^i \cdot \nabla v^j \, d\mathbf{x} + \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \alpha_\ell^j y_\ell^j v^j \, dS \\ &+ \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \alpha_\ell^j(\mu_\ell^j - \sigma^j) v^j \, dS = \sum_{\ell=1}^M \int_{\Omega_\ell} \mathcal{K}(\mathbf{y}_\ell) \cdot \mathbf{v} \, d\mathbf{x} \end{aligned}$$

holds for every  $\mathbf{v} \in \mathbf{W}^{1,2}$  and almost every  $t \in (0, T)$ .

Before proceeding to the proof of the main result of the paper, we prepare some auxiliary lemmas and propositions.

For a fixed  $R > 0$  define the closed ball  $B_R(T) \subset \mathcal{X}_{\ell,T}$

$$B_R(T) := \{\phi \in \mathcal{X}_{\ell,T}; \|\phi\|_{\mathcal{X}_{\ell,T}} \leq R\}.$$

**Lemma 5.2.** *Let  $\phi_\ell \in B_R(T)$ . Then*

$$\|\mathcal{K}(\phi_\ell)\|_{\mathcal{Y}_{\ell,T}} \leq c_1 C(T) \left( \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^3 + \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2 + \|\phi_\ell\|_{\mathcal{X}_{\ell,T}} \right) + c_2, \quad (28)$$

where the function  $C(T) \rightarrow 0_+$  for  $T \rightarrow 0_+$  and the constants  $c_1, c_2 > 0$ , both independent of  $\phi_\ell$ , do not depend on  $T$ .

*Proof.* The proof is rather technical. To derive the estimate (28) we extensively use the embeddings and estimates (7)–(12). First, for all  $\phi_\ell \in B_R(T)$  we have

$$\begin{aligned} \|\mathcal{K}(\phi_\ell)\|_{\mathcal{Y}_{\ell,T}} \leq & \left\| \left( \widehat{\mathbf{b}}_\ell(\mathbf{0}) - \widehat{\mathbf{b}}_\ell(\phi_\ell) \right) \phi'_\ell(t) \right\|_{\mathcal{Y}_{\ell,T}} + \left\| \nabla \cdot [\widehat{a}_\ell^{ji}(\mathbf{0}) - \widehat{a}_\ell^{ji}(\phi_\ell)] \nabla \phi_\ell^i \right\|_{\mathcal{Y}_{\ell,T}} \\ & + \left\| \nabla \cdot [\widehat{a}_\ell^{ji}(\phi_\ell) \nabla \mu_\ell^i] \right\|_{\mathcal{Y}_{\ell,T}} + \left\| \widehat{\mathbf{f}}_\ell(\phi_\ell) \right\|_{\mathcal{Y}_{\ell,T}}. \end{aligned} \quad (29)$$

Now, we have to estimate each term on the right-hand side of (29). Successively, we use (9) and (A1) (see Subsection 3.1) to estimate the first term:

$$\begin{aligned} \left\| \left( \widehat{\mathbf{b}}_\ell(\mathbf{0}) - \widehat{\mathbf{b}}_\ell(\phi_\ell) \right) \phi'_\ell(t) \right\|_{\mathcal{Y}_{\ell,T}} & \leq \left\| \left( \widehat{\mathbf{b}}_\ell(\mathbf{0}) - \widehat{\mathbf{b}}_\ell(\phi_\ell) \right) \phi''_\ell(t) \right\|_{L^2(Q_{\ell T})^2} \\ & \quad + \left\| \frac{\partial \widehat{b}_\ell^{ji}(\phi_\ell)}{\partial \phi_\ell^l} (\phi_\ell^l)'(t) (\phi_\ell^i)'(t) \right\|_{L^2(Q_{\ell T})^2} \\ & \leq c_1 \|\phi_\ell\|_{L^\infty(Q_{\ell T})^2} \|\phi''_\ell(t)\|_{L^2(Q_{\ell T})^2} \\ & \quad + c_2 \|\phi'_\ell(t)\|_{L^4(Q_{\ell T})^2} \\ & \leq c_1 T^{1/2} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2 + c_2 T^{1/4} \|\phi'_\ell(t)\|_{L^8(0,T;\mathbf{L}_\ell^4)}^2 \\ & \leq c_1 T^{1/2} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2 + c_2 T^{1/4} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2. \end{aligned} \quad (30)$$

Similarly, estimating the second term in (29) in the norm of the space  $\mathcal{Y}_{\ell,T}$

we arrive at

$$\begin{aligned}
\|\nabla \cdot [(\widehat{a}_\ell^{ji}(\mathbf{0}) - \widehat{a}_\ell^{ji}(\boldsymbol{\phi}_\ell)) \nabla \phi_\ell^i]\|_{\mathcal{Y}_{\ell,T}} &\leq \left\| \frac{\partial^2 \widehat{a}_\ell^{ji}(\boldsymbol{\phi}_\ell)}{\partial \phi_\ell^l \partial \phi_\ell^r} (\phi_\ell^r)'(t) \nabla \phi_\ell^l \cdot \nabla \phi_\ell^i \right\|_{L^2(Q_{\ell T})^2} \\
&+ \left\| \frac{\partial \widehat{a}_\ell^{ji}(\boldsymbol{\phi}_\ell)}{\partial \phi_\ell^l} \nabla [(\phi_\ell^l)'(t)] \cdot \nabla \phi_\ell^i \right\|_{L^2(Q_{\ell T})^2} \\
&+ \left\| \frac{\partial^2 \widehat{a}_\ell^{ji}(\mathbf{x}, \boldsymbol{\phi}_\ell)}{\partial x^k \partial \phi_\ell^l} (\phi_\ell^l)'(t) \frac{\partial \phi_\ell^i}{\partial x^k} \right\|_{L^2(Q_{\ell T})^2} \\
&+ \left\| \frac{\partial \widehat{a}_\ell^{ji}(\boldsymbol{\phi}_\ell)}{\partial \phi_\ell^l} (\phi_\ell^l)'(t) \Delta \phi_\ell^i \right\|_{L^2(Q_{\ell T})^2} \\
&+ \|\widehat{\mathbf{a}}_\ell(\mathbf{0}) - \widehat{\mathbf{a}}_\ell(\boldsymbol{\phi}_\ell)\| \Delta \phi_\ell'(t) \|_{L^2(Q_{\ell T})^2} \\
&+ \|\nabla [\widehat{a}_\ell^{ji}(\mathbf{0}) - \widehat{a}_\ell^{ji}(\mathbf{x}, \boldsymbol{\phi}_\ell)] \cdot \nabla (\phi_\ell^i)'\|_{L^2(Q_{\ell T})^2}.
\end{aligned} \tag{31}$$

The first integral on the right hand side in (31) can be estimated

$$\begin{aligned}
\left\| \frac{\partial^2 \widehat{a}_\ell^{ji}(\boldsymbol{\phi}_\ell)}{\partial \phi_\ell^l \partial \phi_\ell^r} (\phi_\ell^r)'(t) \nabla \phi_\ell^l \cdot \nabla \phi_\ell^i \right\|_{L^2(Q_{\ell T})^2}^2 &\leq c \int_0^T \left( \int_{\Omega_\ell} |\phi_\ell'(t)|^2 |\nabla \phi_\ell|^4 d\mathbf{x} \right) dt \\
&\leq c \int_0^T \|\phi_\ell'(t)\|_{\mathbf{L}_\ell^4}^2 \|\phi_\ell\|_{\mathbf{W}_\ell^{1,8}}^4 dt \\
&\leq c \|\phi_\ell'(t)\|_{L^2(0,T;\mathbf{L}_\ell^4)}^2 \|\phi_\ell\|_{L^\infty(0,T;\mathbf{W}_\ell^{1,8})}^4 \\
&\leq c T^{3/4} \|\phi_\ell'(t)\|_{L^8(0,T;\mathbf{L}_\ell^4)}^2 T^{1/2} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^4
\end{aligned}$$

and applying (9) we obtain

$$\left\| \frac{\partial^2 \widehat{a}_\ell^{ji}(\boldsymbol{\phi}_\ell)}{\partial \phi_\ell^l \partial \phi_\ell^r} (\phi_\ell^r)'(t) \nabla \phi_\ell^l \cdot \nabla \phi_\ell^i \right\|_{L^2(Q_{\ell T})^2} \leq c T^{5/8} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^3. \tag{32}$$

Similarly

$$\begin{aligned}
\left\| \frac{\partial \widehat{a}_\ell^{ji}(\boldsymbol{\phi}_\ell)}{\partial \phi_\ell^l} \nabla [(\phi_\ell^l)'(t)] \cdot \nabla \phi_\ell^i \right\|_{L^2(Q_{\ell T})^2}^2 &\leq c \int_0^T \left( \int_{\Omega_\ell} |\nabla \phi_\ell'(t)|^2 |\nabla \phi_\ell|^2 d\mathbf{x} \right) dt \\
&\leq c \int_0^T \|\phi_\ell'(t)\|_{\mathbf{W}_\ell^{1,3}}^2 \|\phi_\ell\|_{\mathbf{W}_\ell^{1,6}}^2 dt \\
&\leq c \|\phi_\ell'(t)\|_{L^2(0,T;\mathbf{W}_\ell^{1,3})}^2 \|\phi_\ell\|_{L^\infty(0,T;\mathbf{W}_\ell^{1,6})}^2 \\
&\leq c T^{1/3} \|\phi_\ell'(t)\|_{L^3(0,T;\mathbf{W}_\ell^{1,3})}^2 T^{1/2} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2
\end{aligned}$$

and using (12) we get

$$\left\| \frac{\partial \widehat{a}_\ell^{ji}(\phi_\ell)}{\partial \phi_\ell^l} \nabla[(\phi_\ell^l)'(t)] \cdot \nabla \phi_\ell^i \right\|_{L^2(Q_{\ell T})^2} \leq c T^{5/12} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2. \quad (33)$$

Similarly, the third term in (31) can be estimated as

$$\left\| \frac{\partial^2 \widehat{a}_\ell^{ji}(\mathbf{x}, \phi_\ell)}{\partial x^k \partial \phi_\ell^l} (\phi_\ell^l)'(t) \frac{\partial \phi_\ell^i}{\partial x^k} \right\|_{L^2(Q_{\ell T})^2} \leq c T^{5/12} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2. \quad (34)$$

Further

$$\begin{aligned} \left\| \frac{\partial \widehat{a}_\ell^{ji}(\phi_\ell)}{\partial \phi_\ell^l} (\phi_\ell^l)'(t) \Delta \phi_\ell^i \right\|_{L^2(Q_{\ell T})^2}^2 &\leq c \int_0^T \left( \int_{\Omega_\ell} |\phi_\ell'(t)|^2 |\Delta \phi_\ell|^2 d\mathbf{x} \right) dt \\ &\leq c \int_0^T \|\phi_\ell'(t)\|_{\mathbf{L}_\ell^\infty}^2 \|\phi_\ell\|_{\mathbf{W}_\ell^{2,2}}^2 dt \\ &\leq c \|\phi_\ell'(t)\|_{L^2(0,T;\mathbf{L}_\ell^\infty)}^2 \|\phi_\ell\|_{L^\infty(0,T;\mathbf{W}_\ell^{2,2})}^2 \\ &\leq c T^{1/3} \|\phi_\ell'(t)\|_{L^3(0,T;\mathbf{L}_\ell^\infty)}^2 T^{1/2} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2. \end{aligned} \quad (35)$$

Now (35) and (12) imply

$$\left\| \frac{\partial \widehat{a}_\ell^{ji}(\phi_\ell)}{\partial \phi_\ell^l} (\phi_\ell^l)'(t) \Delta \phi_\ell^i \right\|_{L^2(Q_{\ell T})^2} \leq c T^{5/12} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2. \quad (36)$$

(10) yields the estimate

$$\begin{aligned} \|[\widehat{\mathbf{a}}_\ell(\mathbf{0}) - \widehat{\mathbf{a}}_\ell(\phi_\ell)] \Delta \phi_\ell'(t)\|_{L^2(Q_{\ell T})^2} &\leq c \|\phi_\ell\|_{L^\infty(Q_{\ell T})^2} \|\Delta \phi_\ell'(t)\|_{L^2(Q_{\ell T})^2} \\ &\leq c T^{1/2} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}} \|\Delta \phi_\ell'(t)\|_{L^2(Q_{\ell T})^2} \\ &\leq c T^{1/2} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2, \end{aligned} \quad (37)$$

where we have used the Lipschitz continuity of  $\widehat{\mathbf{a}}_\ell$ . In the similar way one can deduce

$$\begin{aligned} &\|\nabla[\widehat{a}_\ell^{ji}(\mathbf{0}) - \widehat{a}_\ell^{ji}(\phi_\ell)] \cdot \nabla(\phi_\ell^i)'(t)\|_{L^2(Q_{\ell T})^2} \\ &\leq c_1 \|\nabla \phi_\ell'(t)\|_{L^2(0,T;\mathbf{L}_\ell^2)} + c_2 \|\phi_\ell\|_{L^\infty(0,T;\mathbf{W}_\ell^{1,6})} \|\phi_\ell'(t)\|_{L^2(0,T;\mathbf{W}_\ell^{1,3})} \\ &\leq c_1 T^{1/6} \|\phi_\ell'(t)\|_{L^3(0,T;\mathbf{W}_\ell^{1,3})} + c_2 T^{1/4} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}} T^{1/6} \|\phi_\ell'(t)\|_{L^3(0,T;\mathbf{W}_\ell^{1,3})} \\ &\leq c_1 T^{1/6} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}} + c_2 T^{5/12} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2. \end{aligned} \quad (38)$$

Finally, the estimates (31)–(38) imply

$$\begin{aligned} \|\nabla \cdot [\widehat{a}_\ell^{ji}(\mathbf{0}) - \widehat{a}_\ell^{ji}(\phi_\ell)] \nabla \phi_\ell^i\|_{\mathcal{Y}_{\ell,T}} \\ \leq cC(T) \left( \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^3 + \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2 + \|\phi_\ell\|_{\mathcal{X}_{\ell,T}} \right), \end{aligned} \quad (39)$$

where the function  $C(T) \rightarrow 0_+$  for  $T \rightarrow 0_+$  and  $c$  is independent of  $T$ .

Further, estimating the third term on the right hand side in (29) one obtains

$$\begin{aligned} \|\nabla \cdot [\widehat{a}_\ell^{ji}(\phi_\ell) \nabla \mu_\ell^i]\|_{\mathcal{Y}_{\ell,T}} &\leq \left\| \frac{\partial^2 \widehat{a}_\ell^{ji}(\phi_\ell)}{\partial \phi_\ell^l \partial \phi_\ell^r} (\phi_\ell^r)'(t) \nabla \phi_\ell^l \cdot \nabla \mu_\ell^i \right\|_{L^2(Q_{\ell T})^2} \\ &+ \left\| \frac{\partial \widehat{a}_\ell^{ji}(\phi_\ell)}{\partial \phi_\ell^l} \nabla [(\phi_\ell^l)'(t)] \cdot \nabla \mu_\ell^i \right\|_{L^2(Q_{\ell T})^2} \\ &+ \left\| \frac{\partial \widehat{a}_\ell^{ji}(\phi_\ell)}{\partial \phi_\ell^l} (\phi_\ell^l)'(t) \Delta \mu_\ell^i \right\|_{L^2(Q_{\ell T})^2} \\ &+ \left\| \frac{\partial^2 \widehat{a}_\ell^{ji}(\mathbf{x}, \phi_\ell)}{\partial x^k \partial \phi_\ell^l} (\phi_\ell^l)'(t) \frac{\partial \mu_\ell^i}{\partial x^k} \right\|_{L^2(Q_{\ell T})} \\ &+ \|\nabla \cdot [\widehat{a}_\ell^{ji}(\mathbf{0}) \nabla \mu_\ell^i]\|_{\mathbf{W}_\ell^{1,2}}. \end{aligned} \quad (40)$$

Estimating each term on the right hand side we arrive at

$$\begin{aligned} \left\| \frac{\partial^2 \widehat{a}_\ell^{ji}(\phi_\ell)}{\partial \phi_\ell^l \partial \phi_\ell^r} (\phi_\ell^r)'(t) \nabla \phi_\ell^l \cdot \nabla \mu_\ell^i \right\|_{L^2(Q_{\ell T})^2}^2 &\leq c \int_0^T \|\phi_\ell'(t)\|_{\mathbf{L}_\ell^4}^2 \|\nabla \phi_\ell\|_{\mathbf{L}_\ell^8}^2 \|\nabla \mu_\ell\|_{\mathbf{L}_\ell^8}^2 dt \\ &\leq cT^{3/4} \|\phi_\ell'(t)\|_{L^8(0,T;\mathbf{L}_\ell^4)}^2 \|\phi_\ell\|_{L^\infty(0,T;\mathbf{W}_\ell^{1,8})}^2 \|\nabla \mu_\ell\|_{\mathbf{L}_\ell^8}^2 \\ &\leq cT^{5/4} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^4, \end{aligned} \quad (41)$$

where we have used the estimate  $\|\phi_\ell\|_{L^\infty(0,T;\mathbf{W}_\ell^{1,8})}^2 \leq T^{1/2} \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2$ . Further

$$\begin{aligned} \left\| \frac{\partial \widehat{a}_\ell^{ji}(\phi_\ell)}{\partial \phi_\ell^l} \nabla [(\phi_\ell^l)'(t)] \cdot \nabla \mu_\ell^i \right\|_{L^2(Q_{\ell T})^2}^2 &\leq c \int_0^T \|\nabla \phi_\ell'(t)\|_{\mathbf{L}_\ell^3}^2 \|\nabla \mu_\ell\|_{\mathbf{L}_\ell^6}^2 dt \\ &\leq cT^{1/3} \|\phi_\ell'(t)\|_{L^3(0,T;\mathbf{W}_\ell^{1,3})}^2 \|\nabla \mu_\ell\|_{\mathbf{L}_\ell^6}^2, \end{aligned} \quad (42)$$

$$\begin{aligned}
\left\| \frac{\partial \widehat{a}_\ell^{ji}(\boldsymbol{\phi}_\ell)}{\partial \phi_\ell^l} (\phi_\ell^l)'(t) \Delta \mu_\ell^i \right\|_{L^2(Q_{\ell T})^2}^2 &\leq c \int_0^T \|\boldsymbol{\phi}'_\ell(t)\|_{\mathbf{L}_\ell^\infty}^2 \|\Delta \boldsymbol{\mu}_\ell\|_{\mathbf{L}_\ell^2}^2 dt \\
&\leq c T^{1/3} \|\boldsymbol{\phi}'_\ell(t)\|_{L^3(0,T;\mathbf{L}_\ell^\infty)}^2 \|\Delta \boldsymbol{\mu}_\ell\|_{\mathbf{L}_\ell^2}^2.
\end{aligned} \tag{43}$$

and finally

$$\begin{aligned}
\left\| \frac{\partial^2 \widehat{a}_\ell^{ji}(\mathbf{x}, \boldsymbol{\phi}_\ell)}{\partial x^k \partial \phi_\ell^l} (\phi_\ell^l)'(t) \frac{\partial \mu_\ell^i}{\partial x^k} \right\|_{L^2(Q_{\ell T})^2}^2 &\leq c \int_0^T \|\boldsymbol{\phi}'_\ell(t)\|_{\mathbf{L}_\ell^\infty}^2 \|\nabla \boldsymbol{\mu}_\ell\|_{\mathbf{L}_\ell^2}^2 dt \\
&\leq c T^{1/3} \|\boldsymbol{\phi}'_\ell(t)\|_{L^3(0,T;\mathbf{L}_\ell^\infty)}^2 \|\nabla \boldsymbol{\mu}_\ell\|_{\mathbf{L}_\ell^2}^2.
\end{aligned} \tag{44}$$

The inequalities (40)–(44) yield the estimate

$$\|\nabla \cdot [\widehat{a}_\ell^{ji}(\boldsymbol{\phi}_\ell) \nabla \mu_\ell^i]\|_{\mathcal{Y}_{\ell,T}} \leq c_1 T^{5/8} \|\boldsymbol{\phi}_\ell\|_{\mathcal{X}_{\ell,T}}^2 + c_2 T^{1/6} \|\boldsymbol{\phi}_\ell\|_{\mathcal{X}_{\ell,T}} + c_3, \tag{45}$$

where the constant  $c_3$  in (45) bounds the last term in (40). Finally, taking into account (A3), the source term can be estimated as

$$\begin{aligned}
\|\widehat{\mathbf{f}}_\ell(\boldsymbol{\phi}_\ell)\|_{\mathcal{Y}_{\ell,T}} &= \left\| \frac{\partial \widehat{f}_\ell^j(\boldsymbol{\phi}_\ell)}{\partial \phi_\ell^l} (\phi_\ell^l)'(t) \right\|_{L^2(Q_{\ell T})^2} + \|\widehat{\mathbf{f}}_\ell(\mathbf{0})\|_{\mathbf{W}_\ell^{1,2}} \\
&\leq c T^{1/6} \|\boldsymbol{\phi}'_\ell\|_{L^3(0,T;\mathbf{L}_\ell^2)} + c_2 \\
&\leq c T^{1/6} \|\boldsymbol{\phi}_\ell\|_{\mathcal{X}_{\ell,T}} + c_2.
\end{aligned} \tag{46}$$

Altogether, the estimates (30), (39), (45) and (46) yield the inequality (28).  $\square$

**Lemma 5.3.** *There exists a nondecreasing function  $c(R)$  ( $c(R)$  does not depend on  $T$ ,  $\boldsymbol{\phi}_\ell$  and  $\widetilde{\boldsymbol{\phi}}_\ell$ ) such that for all  $\boldsymbol{\phi}_\ell, \widetilde{\boldsymbol{\phi}}_\ell \in B_R(T)$*

$$\|\mathcal{K}(\boldsymbol{\phi}_\ell) - \mathcal{K}(\widetilde{\boldsymbol{\phi}}_\ell)\|_{\mathcal{Y}_{\ell,T}} \leq c(R) C(T) \|\boldsymbol{\phi}_\ell - \widetilde{\boldsymbol{\phi}}_\ell\|_{\mathcal{X}_{\ell,T}}, \tag{47}$$

where the function  $C(T) \rightarrow 0_+$  for  $T \rightarrow 0_+$ .

*Sketch of the proof.* Similarly to Lemma 5.2, the proof is rather technical. Therefore, we only sketch the procedure and omit the detailed derivations.



First, we estimate

$$\begin{aligned}
\|\mathcal{K}(\phi_\ell) - \mathcal{K}(\tilde{\phi}_\ell)\|_{\mathcal{Y}_{\ell,T}} &\leq \left\| \left( \widehat{\mathbf{b}}_\ell(\mathbf{0}) - \widehat{\mathbf{b}}_\ell(\phi_\ell) \right) \phi'_\ell(t) - \left( \widehat{\mathbf{b}}_\ell(\mathbf{0}) - \widehat{\mathbf{b}}_\ell(\tilde{\phi}_\ell) \right) \tilde{\phi}'_\ell(t) \right\|_{\mathcal{Y}_{\ell,T}} \\
&\quad + \left\| \nabla \cdot \left[ \left( \widehat{a}_\ell^{ji}(\mathbf{0}) - \widehat{a}_\ell^{ji}(\phi_\ell) \right) \nabla \phi_\ell^i \right] - \nabla \cdot \left[ \left( \widehat{a}_\ell^{ji}(\mathbf{0}) - \widehat{a}_\ell^{ji}(\tilde{\phi}_\ell) \right) \nabla \tilde{\phi}_\ell^i \right] \right\|_{\mathcal{Y}_{\ell,T}} \\
&\quad + \left\| \nabla \cdot \left[ \widehat{a}_\ell^{ji}(\phi_\ell) \nabla \mu_\ell^i \right] - \nabla \cdot \left[ \widehat{a}_\ell^{ji}(\tilde{\phi}_\ell) \nabla \mu_\ell^i \right] \right\|_{\mathcal{Y}_{\ell,T}} + \left\| \widehat{\mathbf{f}}_\ell(\phi_\ell) - \widehat{\mathbf{f}}_\ell(\tilde{\phi}_\ell) \right\|_{\mathcal{Y}_{\ell,T}}.
\end{aligned} \tag{48}$$

The right hand side in (48) can be further estimated by

$$\begin{aligned}
&\left\| \widehat{\mathbf{b}}_\ell(\mathbf{0}) \left( \phi'_\ell(t) - \tilde{\phi}'_\ell(t) \right) \right\|_{\mathcal{Y}_{\ell,T}} + \left\| \left( \widehat{\mathbf{b}}_\ell(\phi_\ell) - \widehat{\mathbf{b}}_\ell(\tilde{\phi}_\ell) \right) \phi'_\ell(t) \right\|_{\mathcal{Y}_{\ell,T}} \\
&\quad + \left\| \widehat{\mathbf{b}}_\ell(\tilde{\phi}_\ell) \left( \phi'_\ell(t) - \tilde{\phi}'_\ell(t) \right) \right\|_{\mathcal{Y}_{\ell,T}} + \left\| \nabla \cdot \left[ \widehat{a}_\ell^{ji}(\mathbf{0}) \nabla \left( \phi_\ell^i - \tilde{\phi}_\ell^i \right) \right] \right\|_{\mathcal{Y}_{\ell,T}} \\
&\quad + \left\| \nabla \cdot \left[ \widehat{a}_\ell^{ji}(\phi_\ell) \nabla \left( \phi_\ell^i - \tilde{\phi}_\ell^i \right) \right] \right\|_{\mathcal{Y}_{\ell,T}} + \left\| \nabla \cdot \left[ \left( \widehat{a}_\ell^{ji}(\phi_\ell) - \widehat{a}_\ell^{ji}(\tilde{\phi}_\ell) \right) \nabla \tilde{\phi}_\ell^i \right] \right\|_{\mathcal{Y}_{\ell,T}} \\
&\quad + \left\| \nabla \cdot \left[ \left( \widehat{a}_\ell^{ji}(\phi_\ell) - \widehat{a}_\ell^{ji}(\tilde{\phi}_\ell) \right) \nabla \mu_\ell^i \right] \right\|_{\mathcal{Y}_{\ell,T}} + \left\| \widehat{\mathbf{f}}_\ell(\phi_\ell) - \widehat{\mathbf{f}}_\ell(\tilde{\phi}_\ell) \right\|_{\mathcal{Y}_{\ell,T}}.
\end{aligned} \tag{49}$$

Estimating each term in (49) using the same arguments as in the proof of Lemma 5.2 and the assumptions (A1)–(A3), see Section 3.1, one obtains the inequality

$$\|\mathcal{K}(\phi_\ell) - \mathcal{K}(\tilde{\phi}_\ell)\|_{\mathcal{Y}_{\ell,T}} \leq \underbrace{c_1 (R^2 + R + 1)}_{c(R)} C(T) \|\phi_\ell - \tilde{\phi}_\ell\|_{\mathcal{X}_{\ell,T}} \tag{50}$$

for all  $\phi_\ell, \tilde{\phi}_\ell \in B_R(T)$ . Now (50) yields (47).  $\square$

Using Definition 6 and Theorem 4.1 we can formulate the following

**Definition 7.** Let  $\mathcal{L} : \mathcal{X}_{\ell,T} \rightarrow \mathcal{X}_{\ell,T}$  be an operator such that  $\mathcal{L}(\phi_\ell) = \mathbf{y}_\ell$ , if and only if

$$\begin{aligned}
&\sum_{\ell=1}^M \int_{\Omega_\ell} \widehat{b}_\ell^{ji}(\mathbf{0}) \frac{\partial y_\ell^i}{\partial t} v^j \, d\mathbf{x} + \sum_{\ell=1}^M \int_{\Omega_\ell} \widehat{a}_\ell^{ji}(\mathbf{0}) \nabla y_\ell^i \cdot \nabla v^j \, d\mathbf{x} + \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \alpha_\ell^j y_\ell^j v^j \, dS \\
&\quad + \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \alpha_\ell^j (\mu_\ell^j - \sigma^j) v^j \, dS = \sum_{\ell=1}^M \int_{\Omega_\ell} \mathcal{K}(\phi_\ell) \cdot \mathbf{v} \, d\mathbf{x}
\end{aligned}$$

holds for every  $\mathbf{v} \in \mathbf{W}^{1,2}$  and almost every  $t \in (0, T)$  and  $\mathbf{y}_\ell(\mathbf{x}, 0) = \mathbf{0}$  in  $\Omega_\ell$ .

*Proof of Theorem 5.1 (Main result).* The proof of the main result is based on the Banach fixed point theorem. Lemma 5.2 and the estimate (24) imply the inequality

$$\begin{aligned}\|\mathcal{L}(\phi_\ell)\|_{\mathcal{X}_{\ell,T}} &\leq c_1 \|\mathcal{K}(\phi_\ell)\|_{\mathcal{Y}_{\ell,T}} + K_1 \\ &\leq c_2 C(T) \left( \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^3 + \|\phi_\ell\|_{\mathcal{X}_{\ell,T}}^2 + \|\phi_\ell\|_{\mathcal{X}_{\ell,T}} \right) + K_2\end{aligned}\quad (51)$$

for all  $\phi_\ell \in B_R(T)$ , where  $K_1$  and  $K_2$  are positive nondecreasing functions with respect to  $T$  and independent of  $\phi_\ell$  and the constants  $c_1, c_2$  are independent of  $\phi_\ell$  and  $T$ . Further, linearity of (16)–(20), the estimate (24) and Lemma 5.3 imply

$$\begin{aligned}\|\mathcal{L}(\phi_\ell) - \mathcal{L}(\tilde{\phi}_\ell)\|_{\mathcal{X}_{\ell,T}} &\leq c_1 \|\mathcal{K}(\phi_\ell) - \mathcal{K}(\tilde{\phi}_\ell)\|_{\mathcal{Y}_{\ell,T}} \\ &\leq c(R) C(T) \|\phi_\ell - \tilde{\phi}_\ell\|_{\mathcal{X}_{\ell,T}} \quad \text{for all } \phi_\ell, \tilde{\phi}_\ell \in B_R(T),\end{aligned}\quad (52)$$

where  $c(R)$  is some nondecreasing function and  $C(T) \rightarrow 0_+$  for  $T \rightarrow 0_+$ . Now (51) and (52) imply that for sufficiently small  $T^* \in (0, T]$  there exists  $R > 0$  such that  $\mathcal{L} : \mathcal{X}_{\ell,T^*} \rightarrow \mathcal{X}_{\ell,T^*}$  maps  $B_R(T^*)$  into itself and  $\mathcal{L}$  is a strict contraction in  $B_R(T^*)$ . Hence, using the contraction mapping principle we have the existence of a fixed point  $\mathbf{y}_\ell \in B_R(T^*) \subset \mathcal{X}_{\ell,T^*}$ , such that  $\mathcal{L}(\mathbf{y}_\ell) = \mathbf{y}_\ell$ .  $\mathbf{y}_\ell$  is uniquely determined in the ball  $B_R(T^*)$ . Set  $\mathbf{u}_\ell = \boldsymbol{\mu}_\ell + \mathbf{y}_\ell$ . By Remark 5 the function  $\mathbf{u}_\ell$  is the strong solution of Problem  $(P_0)$  on  $(0, T^*)$ .  $\square$

## 6. Applications

In this Section, we present examples of the coefficients of the parabolic system (1) related to models of transport in porous media. Note that for brevity, we omit the subscript  $\ell$  and the dependence of all variables on  $\mathbf{x}$  and  $t$  in what follows.

All available engineering models of simultaneous heat and moisture transfer possess a common structure, derived from two balance equations of heat and mass [9]:

$$\frac{dH}{dt} = -\nabla \cdot \mathbf{j}_Q + Q, \quad \frac{dM}{dt} = -\nabla \cdot \mathbf{j}_m, \quad (53)$$

where  $H$  ( $\text{Jm}^{-3}$ ) is the specific enthalpy,  $M$  ( $\text{kgm}^{-3}$ ) denotes the partial moisture density,  $Q$  ( $\text{Jm}^{-3}\text{s}^{-1}$ ) stands for the intensity of internal heat sources and  $\mathbf{j}_Q$  ( $\text{Jm}^{-2}\text{s}^{-1}$ ) and  $\mathbf{j}_m$  ( $\text{kgm}^{-2}\text{s}^{-1}$ ) are the heat and moisture fluxes, respectively. This structure is also reflected in the choice of the unknowns  $\mathbf{u}$ ,

which consist of the temperature  $u^1 = \theta$  (K) and a quantity related to the moisture content.

Individual models are then generated by the choice of the second state variable  $u^2$  and of the individual components in system (53). In Sections 6.1 and 6.2, following the expositions of Dalík *et al.* [4] and Černý and Rovnaníková [3], we briefly introduce two such representatives due to Kiessl [12] and Künzle [18], simplified by assuming that freezing of water in pores has a negligible effect. An interested reader is referred to [3, 4, 19] for additional discussion of the models and to [3] for details on the terminology used hereafter.

### 6.1. The Kiessl model

The enthalpy term in the Kiessl model is postulated in the form

$$H = \rho_0 c_0 \theta + \rho_w c_w w \theta, \quad (54)$$

where  $\rho_0$  (kgm<sup>-3</sup>) and  $c_0$  (Jkg<sup>-1</sup>K<sup>-1</sup>) denote the partial density and the specific heat capacity of the dry porous matrix,  $\rho_w$  and  $c_w$  are analogous quantities for water and  $w$  (-) is the relative moisture content by volume. The heat flux follows from the Fourier law for isotropic materials

$$\mathbf{j}_Q = -\lambda(w, \theta) \nabla \theta, \quad (55)$$

with  $\lambda$  (Jm<sup>-1</sup>s<sup>-1</sup>K<sup>-1</sup>) being the state-dependent coefficient of thermal conductivity.

The moisture balance is based on the moisture density provided by

$$M = \rho_w w + (e - w) \varphi \rho_{p,s}(\theta) \quad (56)$$

where  $e \geq w$  (-) denotes the porosity,  $\varphi$  is the relative humidity and  $\rho_{p,s} \leq \rho_w$  (kgm<sup>-3</sup>) is the material-independent partial density of the saturated vapor phase, given as a smooth increasing function of  $\theta$ . Assuming again isotropy of the material, the corresponding moisture flux is expressed in the form

$$\mathbf{j}_m = -(D_w(w, \theta) \nabla w + D_\varphi(w, \theta) \nabla \varphi + D_\theta(w, \theta) \nabla \theta), \quad (57)$$

where  $D_w$  (kgm<sup>-1</sup>s<sup>-1</sup>),  $D_\varphi$  (kgm<sup>-1</sup>s<sup>-1</sup>) and  $D_\theta$  (kgm<sup>-1</sup>K<sup>-1</sup>s<sup>-1</sup>) denote material-specific diffusion coefficients, which need to be determined experimentally. Finally, the internal heat sources

$$Q = L_v (D_w(w, \theta) \nabla w + D_\varphi(w, \theta) \nabla \varphi - \frac{\partial}{\partial t} [(e - w) \varphi \rho_{p,s}(\theta)]) \quad (58)$$

quantify the influence of phase changes in pores by means of the latent heat of evaporation of water  $L_v$  ( $\text{Jkg}^{-1}$ ).

To close the model, Kiessl in [12] related the auxiliary variables  $w$  and  $\varphi$  to a dimensionless moisture potential  $\Phi = u^2$  via monotone, material-dependent, transformations

$$w = f(\Phi), \quad \varphi = g(\Phi), \quad (59)$$

satisfying  $f(0) = g(0) = 0$  and  $\frac{dg}{d\Phi}(0) = 1$ . In particular,  $f$  denotes the sorption isotherm, whereas  $g$  reflects the pore size distribution. By employing these identifies, the individual coefficients in (1) receive the form (cf. [4])

$$B^1 = \rho_0 c_0 \theta + \rho_w c_w g(\Phi) \theta + L_v (e - f(\Phi)) g(\Phi) \rho_{p,s}(\theta), \quad (60a)$$

$$B^2 = \rho_w f(\Phi) + (e - f(\Phi)) g(\Phi) \rho_{p,s}(\theta), \quad (60b)$$

$$a^{11} = \lambda(f(\Phi), \theta), \quad (60c)$$

$$a^{12} = L_v \left( D_w(f(\Phi), \theta) \frac{df(\Phi)}{d\Phi} + D_\varphi(f(\Phi), \theta) \frac{dg(\Phi)}{d\Phi} \right), \quad (60d)$$

$$a^{21} = D_\theta(f(\Phi), \theta), \quad (60e)$$

$$a^{22} = D_w(f(\Phi), \theta) \frac{df(\Phi)}{d\Phi} + D_\varphi(f(\Phi), \theta) \frac{dg(\Phi)}{d\Phi}. \quad (60f)$$

## 6.2. The Künzel model

In the Künzel framework, the heat balance is described using identical expressions for the enthalpy (54) and the heat flux (55) as previously. In addition, the moisture density is simplified into

$$M = \rho_w w \quad (61)$$

and the moisture flux attains a form

$$\mathbf{j}_m = - \left( \hat{D}_\varphi(\varphi, \theta) \nabla \varphi + \frac{\delta(\theta)}{\mu} \nabla (\varphi p_s(\theta)) \right), \quad (62)$$

in which  $\hat{D}_\varphi$  ( $\text{kgm}^{-1}\text{s}^{-1}$ ) stands for the liquid conduction coefficient,  $\delta$  ( $\text{kgm}^{-1}\text{s}^{-1}\text{Pa}^{-1}$ ) is the vapor diffusion coefficient in air,  $\mu$  (-) is the vapor diffusion resistance factor of a porous material and  $p_s$  (Pa) is the vapor saturation pressure. This yields the heat source term given by

$$Q = L_v \nabla \cdot \left( \frac{\delta(\theta)}{\mu} \nabla (\varphi p_s(\theta)) \right). \quad (63)$$

The relative humidity is chosen as the second unknown,  $u^2 = \varphi$ , and is used to express the associated volumetric moisture content in the form

$$w = h(\varphi), \quad (64)$$

where  $h$  is a monotone moisture storage function with  $h(0) = 0$ . Altogether, the coefficients in system (1) read as

$$B^1 = \rho_0 c_0 \theta + \rho_w c_w h(\varphi) \theta, \quad (65a)$$

$$B^2 = \rho_w h(\varphi), \quad (65b)$$

$$a^{11} = \lambda(h(\varphi), \theta) + L_v \frac{\delta(\theta)}{\mu} \varphi \frac{dp_s(\theta)}{d\theta}, \quad (65c)$$

$$a^{12} = L_v \frac{\delta(\theta)}{\mu} p_s(\theta), \quad (65d)$$

$$a^{21} = \frac{\delta(\theta)}{\mu} \varphi \frac{p_s(\theta)}{d\theta}, \quad (65e)$$

$$a^{22} = D_\varphi(\varphi, \theta) + \frac{\delta(\theta)}{\mu} p_s(\theta). \quad (65f)$$

### 6.3. Structure conditions (A1) and (A2)

The structure conditions (A1)–(A2) closely reflect the physical constraints on the underlying transport models and experimental observations. Concretely, the model parameters (such as e.g.  $f(\Phi)$ ,  $g(\Phi)$  and  $\rho_{p,s}(\theta)$  in the Kiessl model, or  $h(\varphi)$  and  $p_s(\theta)$  in the Künzel model) are obtained by fitting smooth functions to experimental data, determined for a limited range of state variables. The required regularity and boundedness of coefficients  $\mathbf{B}$ ,  $a^{jk}$  and positivity of  $a^{jk}$  is therefore ensured. The increasing character of  $\mathbf{B}$  is consistent with the fact that both the specific enthalpy  $H$  and the moisture density  $M$  increase with an increasing temperature and the moisture-related variable, respectively. The ellipticity condition (15) is satisfied due to the fact that the Soret- and Dufour-type fluxes, quantified by  $a^{12}$  and  $a^{21}$ , are dominated by the diagonal contributions  $a^{11}$  and  $a^{22}$ , see also [5]. Therefore, any physically correct form of  $a^{jk}$  must meet this condition. The validity of the assumption (14) then follows from the same physical reasoning.

## Appendix A. Proof of Lemma 4.2

Discretize (23) in time and replace  $\mathbf{u}'_\ell(t_n)$  by the backward difference quotient  $\partial_t^{-h}(\mathbf{w}_\ell)_n = [(\mathbf{w}_\ell)_n - (\mathbf{w}_\ell)_{n-1}]/h$ , where  $h > 0$  is a time step. Suppose  $r = T/h$  is an integer. For simplicity, let us write  $\mathbf{w}_\ell = (\mathbf{w}_\ell)_n$ ,  $\mathbf{f}_\ell = (\mathbf{f}_\ell)_n$ .

We have to solve, successively for  $n = 1, \dots, r$ , the steady problems

$$\begin{aligned} \sum_{\ell=1}^M \int_{\Omega_\ell} \beta_\ell^{ji} \partial_t^{-h} w_\ell^i v^j \, d\mathbf{x} + \sum_{\ell=1}^M \int_{\Omega_\ell} \kappa_\ell^{ji} \nabla w_\ell^i \cdot \nabla v^j \, d\mathbf{x} \\ + \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \nu_\ell^j w_\ell^j v^j \, dS = \sum_{\ell=1}^M \int_{\Omega_\ell} f_\ell^j v^j \, d\mathbf{x} \quad (\text{A.1}) \end{aligned}$$

which hold for every  $\mathbf{v} \in \mathbf{W}^{1,2}$  and  $(\mathbf{w}_\ell)_0 = \mathbf{0}$  in  $\Omega_\ell$ . Test (A.1) by  $[v^1, v^2] = [\kappa_\ell^{21} \varphi^1, \kappa_\ell^{12} \varphi^2]$  and define the bilinear form  $\mathfrak{A}(\mathbf{w}, \boldsymbol{\varphi})$ ;

$$\begin{aligned} \mathfrak{A}(\mathbf{w}_\ell, \boldsymbol{\varphi}) = \frac{1}{h} \sum_{\ell=1}^M \int_{\Omega_\ell} \kappa_\ell^{pj} \beta_\ell^{ji} w_\ell^i \varphi^j \, d\mathbf{x} + \sum_{\ell=1}^M \int_{\Omega_\ell} \kappa_\ell^{pj} \kappa_\ell^{ji} \nabla w_\ell^i \cdot \nabla \varphi^j \, d\mathbf{x} \\ + \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \kappa_\ell^{pj} \nu_\ell^j w_\ell^j \varphi^j \, dS, \quad p = 1, 2, \, p \neq j, \end{aligned}$$

for every  $\boldsymbol{\varphi} \in \mathbf{W}^{1,2}$ . Set  $\boldsymbol{\varphi} = \mathbf{w}_\ell$ . Now, (21), (22) and the Friedrichs inequality yield the  $\mathbf{W}^{1,2}$ -ellipticity, i.e. there exists  $c > 0$  such that

$$c \|\mathbf{w}_\ell\|_{\mathbf{W}_\ell^{1,2}}^2 \leq |\mathfrak{A}(\mathbf{w}_\ell, \mathbf{w}_\ell)| \quad (\text{A.2})$$

for all  $\mathbf{w}_\ell \in \mathbf{W}_\ell^{1,2}$ . Using the Hölder inequality and the standard trace theorem one obtains the continuity of  $\mathfrak{A}$ , i.e. the inequality

$$|\mathfrak{A}(\mathbf{w}_\ell, \mathbf{z}_\ell)| \leq c \|\mathbf{w}_\ell\|_{\mathbf{W}_\ell^{1,2}} \|\mathbf{z}_\ell\|_{\mathbf{W}_\ell^{1,2}} \quad (\text{A.3})$$

which holds for all  $\mathbf{w}_\ell, \mathbf{z}_\ell \in \mathbf{W}_\ell^{1,2}$  and for some positive constant  $c$ . The linearity of  $\mathfrak{A} : \mathbf{W}_\ell^{1,2} \rightarrow (\mathbf{W}_\ell^{1,2})^*$  is obvious. Hence, for every  $\mathbf{f}_\ell \in \mathbf{L}_\ell^2 \subset (\mathbf{W}_\ell^{1,2})^*$  the Lax-Milgram theorem yields the existence of the weak solution  $\mathbf{w}_\ell \in \mathbf{W}_\ell^{1,2}$ . To get higher regularity results (with respect to time), define  $\mathbf{w}_\ell \in \mathbf{W}_\ell^{1,2}$  by (A.1) and test (A.1) by  $[v^1, v^2] = [\kappa_\ell^{21} \partial_t^{-h} w_\ell^1, \kappa_\ell^{12} \partial_t^{-h} w_\ell^2]$  to get

$$\begin{aligned} \sum_{\ell=1}^M \int_{\Omega_\ell} \kappa_\ell^{pj} \beta_\ell^{ji} \partial_t^{-h} w_\ell^i \partial_t^{-h} w_\ell^j \, d\mathbf{x} \\ + \sum_{\ell=1}^M \int_{\Omega_\ell} \kappa_\ell^{pj} \kappa_\ell^{ji} \nabla w_\ell^i \nabla (\partial_t^{-h} w_\ell^j) \, d\mathbf{x} + \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \kappa_\ell^{pj} \nu_\ell^j w_\ell^j \partial_t^{-h} w_\ell^j \, dS \\ = \sum_{\ell=1}^M \int_{\Omega_\ell} \kappa_\ell^{pj} f_\ell^j \partial_t^{-h} w_\ell^j \, d\mathbf{x}, \quad p = 1, 2, \, p \neq j. \quad (\text{A.4}) \end{aligned}$$

Denote by

$$\begin{aligned}\Phi_\ell[\nabla(\mathbf{w}_\ell)_n] &= \kappa_\ell^{21} \kappa_\ell^{11} \frac{1}{2} |\nabla(w_\ell^1)_n|^2 + \kappa_\ell^{12} \kappa_\ell^{22} \frac{1}{2} |\nabla(w_\ell^2)_n|^2 \\ &\quad + \kappa_\ell^{12} \kappa_\ell^{21} \nabla(w_\ell^1)_n \cdot \nabla(w_\ell^2)_n, \quad n = 1, \dots, r. \quad (\text{A.5})\end{aligned}$$

Now we can estimate

$$\Phi'_\ell[(\nabla \mathbf{w}_\ell)_n] \cdot ((\nabla \mathbf{w}_\ell)_n - (\nabla \mathbf{w}_\ell)_{n-1}) \geq \Phi_\ell[(\nabla \mathbf{w}_\ell)_n] - \Phi_\ell[(\nabla \mathbf{w}_\ell)_{n-1}]$$

because  $\Phi_\ell$  is convex. Thus, using Young's inequality, one obtains

$$\begin{aligned}&\sum_{\ell=1}^M \int_{\Omega_\ell} \kappa_\ell^{pj} \beta_\ell^{ji} \partial_t^{-h} w_\ell^i \partial_t^{-h} w_\ell^j \, d\mathbf{x} + \frac{1}{h} \sum_{\ell=1}^M \int_{\Omega_\ell} \Phi_\ell[(\nabla \mathbf{w}_\ell)_n] - \Phi_\ell[(\nabla \mathbf{w}_\ell)_{n-1}] \, d\mathbf{x} \\ &\quad + \frac{1}{h} \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \kappa_\ell^{pj} \nu_\ell^j \left( \frac{1}{2} |(w_\ell^j)_n|^2 - \frac{1}{2} |(w_\ell^j)_{n-1}|^2 \right) \, dS \\ &\leq C(\epsilon) \sum_{\ell=1}^M \|\mathbf{f}_\ell\|_{\mathbf{L}_\ell^2}^2 + \epsilon c_2 \sum_{\ell=1}^M \int_{\Omega_\ell} \|\partial_t^{-h} \mathbf{w}_\ell\|_{\mathbf{L}_\ell^2}^2 \, d\mathbf{x}, \quad p = 1, 2, \, p \neq j, \quad (\text{A.6})\end{aligned}$$

with some arbitrarily small constant  $\epsilon$ . Note that (21) yields the estimate of the parabolic term

$$c \sum_{\ell=1}^M \int_{\Omega_\ell} \|\partial_t^{-h} \mathbf{w}_\ell\|_{\mathbf{L}_\ell^2}^2 \, d\mathbf{x} \leq \sum_{\ell=1}^M \int_{\Omega_\ell} \kappa_\ell^{pj} \beta_\ell^{ji} \partial_t^{-h} w_\ell^i \partial_t^{-h} w_\ell^j \, d\mathbf{x}, \quad p = 1, 2, \, p \neq j,$$

where  $c$  depends on  $\beta_\ell^{ji}$  and  $\kappa_\ell^{ji}$ . Based on the estimate (A.6), the same way as in [22, Proof of Theorem 8.16] we can prove the existence of the solution  $\mathbf{u}_\ell \in L^\infty(0, T; \mathbf{W}_\ell^{1,2}) \hookrightarrow L^\infty(0, T; \mathbf{L}_\ell^2)$  with  $\mathbf{u}'_\ell(t) \in L^2(0, T; \mathbf{L}_\ell^2)$  and the estimate

$$\|\mathbf{u}'_\ell(t)\|_{L^2(0,T;\mathbf{L}_\ell^2)} + \|\mathbf{u}_\ell\|_{L^\infty(0,T;\mathbf{L}_\ell^2)} \leq c \|\mathbf{f}_\ell\|_{L^2(0,T;\mathbf{L}_\ell^2)}. \quad (\text{A.7})$$

Now we can proceed as in [10]. Rewrite the system (16)–(20) in the form

$$\begin{cases} -\nabla \cdot (\kappa_\ell^{ji} \nabla u_\ell^i) &= F_\ell^j := f_\ell^j - \beta_\ell^{ji} \frac{\partial u_\ell^i}{\partial t} & \text{in } Q_{\ell T}, \\ \kappa_\ell^{ji} \frac{\partial u_\ell^i}{\partial \mathbf{n}_\ell} + \nu_\ell^j u_\ell^j &= 0 & \text{on } S_{\ell T}, \\ u_\ell^j &= u_m^j & \text{on } \Gamma_{m\ell} \times (0, T), \\ \kappa_\ell^{ji} \frac{\partial u_\ell^i}{\partial \mathbf{n}_\ell} + \kappa_m^{ji} \frac{\partial u_m^i}{\partial \mathbf{n}_m} &= 0 & \text{on } \Gamma_{m\ell} \times (0, T). \end{cases} \quad (\text{A.8})$$

Since  $\mathbf{u}'(t)_\ell \in L^2(0, T; \mathbf{L}_\ell^2)$  we have  $F_\ell^j(\mathbf{x}, t) \in \mathbf{L}_\ell^2$  for a.e.  $t \in (0, T)$ . According to results for stationary transmission problem (see Appendix C, Corollary 1) we have  $\mathbf{u}_\ell(t) \in \mathbf{W}_\ell^{2,2}$  and the estimate

$$\|\mathbf{u}_\ell(t)\|_{\mathbf{W}_\ell^{2,2}} \leq c \|\mathbf{f}_\ell(t)\|_{\mathbf{L}_\ell^2} \quad (\text{A.9})$$

holds for almost every  $t \in (0, T)$ , where the constant  $c$  is independent of  $t$ . Raising (A.9) and integrating both sides in the inequality (A.9) with respect to time, we get  $\mathbf{u}_\ell \in L^2(0, T; \mathbf{W}_\ell^{2,2})$  and taking into account the estimate (A.7) we arrive at

$$\|\mathbf{u}'_\ell(t)\|_{L^2(0,T;\mathbf{L}_\ell^2)} + \|\mathbf{u}_\ell\|_{L^2(0,T;\mathbf{W}_\ell^{2,2})} + \|\mathbf{u}_\ell\|_{L^\infty(0,T;\mathbf{L}_\ell^2)} \leq c\|\mathbf{f}_\ell\|_{L^2(0,T;\mathbf{L}_\ell^2)}. \quad (\text{A.10})$$

Now taking the derivative of (16)–(20) with respect to time, considering  $\mathbf{f}_\ell \in \mathcal{Y}_{\ell,T}$  (including the compatibility condition on  $\mathbf{f}_\ell(\mathbf{x}, 0)$ ), we conclude  $\mathbf{u}''_\ell(t) \in L^2(0, T; \mathbf{L}_\ell^2)$ ,  $\mathbf{u}'_\ell(t) \in L^2(0, T; \mathbf{W}_\ell^{2,2}) \cap L^\infty(0, T; \mathbf{L}_\ell^2)$  and the estimate (25) follows. The linearity of Problem  $(P_f)$  and the estimate (25) yield the uniqueness.

## Appendix B. Proof of Lemma 4.3

Let  $\mathbf{g}_\ell \in W^{2,2}(0, T; (\mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})^*)$ . We are looking for the solution of the problem defined via

$$\begin{aligned} \sum_{\ell=1}^M \int_{\Omega_\ell} \beta_\ell^{ji} \frac{\partial \xi_\ell^i}{\partial t} v^j \, d\mathbf{x} + \sum_{\ell=1}^M \int_{\Omega_\ell} \kappa_\ell^{ji} \nabla \xi_\ell^i \cdot \nabla v^j \, d\mathbf{x} \\ + \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \nu_\ell^j \xi_\ell^j v^j \, dS = \sum_{\ell=1}^M \langle \mathbf{G}_\ell''(t); \mathbf{v} \rangle_{(\mathbf{W}_\ell^{1,2})^*, \mathbf{W}_\ell^{1,2}} \end{aligned} \quad (\text{B.1})$$

to be satisfied for every  $\mathbf{v} \in \mathbf{W}^{1,2}$ , almost every  $t \in (0, T)$  and  $\xi_\ell(\mathbf{x}, 0) = \mathbf{0}$  in  $\Omega_\ell$ . The duality of  $\langle \mathbf{G}_\ell; \mathbf{v} \rangle_{(\mathbf{W}_\ell^{1,2})^*, \mathbf{W}_\ell^{1,2}}$  corresponds to

$$\langle \mathbf{G}_\ell; \mathbf{v} \rangle_{(\mathbf{W}_\ell^{1,2})^*, \mathbf{W}_\ell^{1,2}} = \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} g_\ell^j v^j \, dS. \quad (\text{B.2})$$

Approximate (B.1) in time by discretization and replace  $\xi'_\ell(t_n)$  by the backward difference quotient  $\partial_t^{-h}(\mathbf{w}_\ell)_n = [(\mathbf{w}_\ell)_n - (\mathbf{w}_\ell)_{n-1}]/h$ , where  $h > 0$  is a time step. Suppose  $r = T/h$  is an integer. Let us write  $\mathbf{w}_\ell = (\mathbf{w}_\ell)_n$  and test (B.1) by  $[v^1, v^2] = [\kappa_\ell^{21}\varphi^1, \kappa_\ell^{12}\varphi^2]$ . We have to solve, successively for  $n = 1, \dots, r$ , the steady problems

$$\begin{aligned} \sum_{\ell=1}^M \int_{\Omega_\ell} \kappa_\ell^{pj} \beta_\ell^{ji} \partial_t^{-h} w_\ell^i \varphi^j \, d\mathbf{x} \\ + \sum_{\ell=1}^M \int_{\Omega_\ell} \kappa_\ell^{pj} \kappa_\ell^{ji} \nabla w_\ell^i \nabla \varphi^j \, d\mathbf{x} + \sum_{\ell=1}^M \int_{\partial\Omega_\ell \cap \Gamma} \kappa_\ell^{pj} \nu_\ell^j w_\ell^j \varphi^j \, dS \\ = \sum_{\ell=1}^M \kappa_\ell^{pj} \langle (G''(t_n))_\ell^j; \varphi^j \rangle_{(\mathbf{W}_\ell^{1,2})^*, \mathbf{W}_\ell^{1,2}}, \quad p = 1, 2, \, p \neq j, \end{aligned} \quad (\text{B.3})$$



to hold for every  $\varphi \in \mathbf{W}^{1,2}$  and  $(\mathbf{w}_\ell)_0 = \mathbf{0}$ . Test (B.3) by  $\varphi = (\mathbf{w}_\ell)_n$  to get the estimate

$$\begin{aligned}
& c_1 \sum_{\ell=1}^M \left( \frac{1}{2} \|(\mathbf{w}_\ell)_n\|_{\mathbf{L}^2}^2 - \frac{1}{2} \|(\mathbf{w}_\ell)_0\|_{\mathbf{L}^2}^2 \right) \\
& + c_2 h \left( \sum_{\ell=1}^M \sum_{m=1}^n \left( \int_{\Omega_\ell} |\nabla (\mathbf{w}_\ell)_m|^2 d\mathbf{x} + \int_{\partial\Omega_\ell \cap \Gamma} |(\mathbf{w}_\ell)_m|^2 dS \right) \right) \\
& \leq \sum_{\ell=1}^M \sum_{m=1}^n \kappa_\ell^{pj} \langle (G''(t_n))_\ell^j, (w_\ell^j)_n \rangle_{(W_\ell^{1,2})^*, W_\ell^{1,2}}, \quad p = 1, 2, p \neq j. \quad (\text{B.4})
\end{aligned}$$

Now we can proceed as in [22, Proof of Lemma 8.6, Proof of Theorem 8.9] to prove the existence of the weak solution  $\boldsymbol{\xi}_\ell \in L^2(0, T; \mathbf{W}_\ell^{1,2})$  (as the limit of Rothe sequences) with the estimate

$$\|\boldsymbol{\xi}_\ell\|_{L^2(0, T; \mathbf{W}_\ell^{1,2})} \leq c \|\mathbf{G}_\ell''(t)\|_{L^2(0, T; (\mathbf{W}_\ell^{1,2})^*)}. \quad (\text{B.5})$$

Let us note that  $\boldsymbol{\xi}_\ell$  stands for  $\mathbf{u}_\ell''(t)$ . Hence  $\mathbf{u}_\ell''(t) \in L^2(0, T; \mathbf{W}_\ell^{1,2}) \hookrightarrow L^2(0, T; \mathbf{L}_\ell^2)$ . Further, let  $\mathbf{g}_\ell \in W^{1,2}(0, T; \mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})$ . The standard theory for parabolic problems yields  $\mathbf{u}_\ell'(t) \in L^\infty(0, T; \mathbf{L}_\ell^2)$ . Using the same procedure as in Appendix A and according to results for stationary transmission problem (see Appendix C, Corollary 1) we conclude  $\mathbf{u}_\ell'(t) \in L^2(0, T; \mathbf{W}_\ell^{2,2}) \cap L^\infty(0, T; \mathbf{L}_\ell^2)$  and (combining with (B.5))

$$\begin{aligned}
& \|\mathbf{u}_\ell''(t)\|_{L^2(0, T; \mathbf{L}_\ell^2)} + \|\mathbf{u}_\ell'(t)\|_{L^2(0, T; \mathbf{W}_\ell^{2,2})} + \|\mathbf{u}_\ell'(t)\|_{L^\infty(0, T; \mathbf{L}_\ell^2)} \\
& \leq c \left( \|\mathbf{g}_\ell\|_{W^{2,2}(0, T; (\mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})^*)} + \|\mathbf{g}_\ell\|_{W^{1,2}(0, T; \mathbf{W}_{\partial\Omega_\ell \cap \Gamma}^{1/2,2})} \right). \quad (\text{B.6})
\end{aligned}$$

The linearity of Problem  $(P_g)$  and the estimate (B.5) yield the uniqueness.

### Appendix C. Transmission problem for elliptic systems in a multi-layer structure

The boundary transmission problem for the elliptic system for the sub-domain  $\Omega_\ell$  is formulated, in the expanded form, as

$$\left\{ \begin{array}{ll} -\nabla \cdot (\varepsilon_\ell^{11}(\mathbf{x}) \nabla u_\ell^1) - \nabla \cdot (\varepsilon_\ell^{12}(\mathbf{x}) \nabla u_\ell^2) &= f_\ell^1 \quad \text{in } \Omega_\ell, \\ -\nabla \cdot (\varepsilon_\ell^{21}(\mathbf{x}) \nabla u_\ell^1) - \nabla \cdot (\varepsilon_\ell^{22}(\mathbf{x}) \nabla u_\ell^2) &= f_\ell^2 \quad \text{in } \Omega_\ell, \\ \varepsilon_\ell^{11}(\mathbf{x}) \frac{\partial u_\ell^1}{\partial \mathbf{n}_\ell} + \varepsilon_\ell^{12}(\mathbf{x}) \frac{\partial u_\ell^2}{\partial \mathbf{n}_\ell} + \alpha_\ell^1 u_\ell^1 &= g_\ell^1 \quad \text{on } \partial\Omega_\ell \cap \Gamma, \\ \varepsilon_\ell^{21}(\mathbf{x}) \frac{\partial u_\ell^1}{\partial \mathbf{n}_\ell} + \varepsilon_\ell^{22}(\mathbf{x}) \frac{\partial u_\ell^2}{\partial \mathbf{n}_\ell} + \alpha_\ell^2 u_\ell^2 &= g_\ell^2 \quad \text{on } \partial\Omega_\ell \cap \Gamma, \\ u_\ell^1 &= u_m^1 \quad \text{on } \Gamma_{m\ell}, \\ u_\ell^2 &= u_m^2 \quad \text{on } \Gamma_{m\ell}, \\ \varepsilon_\ell^{11}(\mathbf{x}) \frac{\partial u_\ell^1}{\partial \mathbf{n}_\ell} + \varepsilon_\ell^{12}(\mathbf{x}) \frac{\partial u_\ell^2}{\partial \mathbf{n}_\ell} + \varepsilon_m^{11}(\mathbf{x}) \frac{\partial u_m^1}{\partial \mathbf{n}_m} + \varepsilon_m^{12}(\mathbf{x}) \frac{\partial u_m^2}{\partial \mathbf{n}_m} &= 0 \quad \text{on } \Gamma_{m\ell}, \\ \varepsilon_\ell^{21}(\mathbf{x}) \frac{\partial u_\ell^1}{\partial \mathbf{n}_\ell} + \varepsilon_\ell^{22}(\mathbf{x}) \frac{\partial u_\ell^2}{\partial \mathbf{n}_\ell} + \varepsilon_m^{21}(\mathbf{x}) \frac{\partial u_m^1}{\partial \mathbf{n}_m} + \varepsilon_m^{22}(\mathbf{x}) \frac{\partial u_m^2}{\partial \mathbf{n}_m} &= 0 \quad \text{on } \Gamma_{m\ell}. \end{array} \right. \quad (\text{C.1})$$

Here we assume that the problem (C.1) is elliptic and has a unique weak solution  $\mathbf{u}_\ell \in \mathbf{W}_\ell^{1,2}$  for  $\mathbf{f}_\ell \in \mathbf{L}_\ell^2$  and  $\mathbf{g}_\ell \in \mathbf{W}_{\ell,\Gamma}^{1/2,2}$ . Further we consider that  $\varepsilon_\ell^{ji}(\mathbf{x})$  are positive Lipschitz continuous functions and  $\alpha_\ell^j$  are prescribed constants.

Elliptic boundary value problems in cornered plane domains are extensively investigated in the literature, see e.g. [6, 13, 14, 15, 17]. The behavior of local solutions of general linear and semilinear transmission problems is studied in [23, 24, 25]. We adapt the general framework stated in the literature to calculate the regularity of the plane transmission problem for the elliptic system of equations (C.1).

It is known (cf. [17, 25]) that, in general, the boundary singularities may occur near corner points at the boundary  $\partial\Omega$ , the points at the boundary where the boundary conditions change their type, the crossing points of interfaces, corner points of inclusions or points, where the interfaces  $\Gamma_{m\ell}$  intersect the exterior boundary of the domain  $\Omega$ . Taking into account the assumptions on admissible domains introduced in Subsection 3.2, only the points where the interfaces  $\Gamma_{m\ell}$  intersect the exterior boundary are of importance in our analysis of vertex singularities. Hence, let  $\mathcal{M}$  be the set of all boundary points  $A \in \Gamma \cap \Gamma_{m\ell}$ ,  $m, \ell = 1, \dots, M$ . As well known, the local regularity is valid outside an arbitrarily small neighborhood of the points  $A \in \mathcal{M}$ . Hence it suffices to prove the regularity for the solution  $\mathbf{u}_\ell$  with small supports. For solutions with arbitrary support the assertion then can be easily proved by means of a partition of unity on  $\Omega$ . Let  $A$  be an arbitrary point from the set  $\mathcal{M}$  and let the support of  $\mathbf{u}_\ell$  be contained in a sufficiently small neighborhood  $\mathcal{U}(A)$  of the point  $A$ . Let  $D$  be a diffeomorphic mapping  $\Omega \cap \mathcal{U}(A)$

$$\eta(|\mathbf{x}|) = \begin{cases} 1 & \text{for } |\mathbf{x}| < \epsilon/2, \\ 0 & \text{for } |\mathbf{x}| > \epsilon. \end{cases}$$

frozen coefficients (now written in compact form using the Einstein summation convention for indices  $i$  and  $j$  running from 1 to 2)

$$\left\{ \begin{array}{ll} -\varepsilon_m^{ji}(\mathbf{0})\Delta w_m^i &= F_m^j \quad \text{in } \mathcal{K}_m, \\ \varepsilon_m^{ji}(\mathbf{0})\frac{\partial w_m^i}{\partial \mathbf{n}_m} &= G_m^j \quad \text{on } \partial\mathcal{K}_m \cap \Gamma, \\ w_\ell^j &= w_{\ell+1}^j \quad \text{on } \partial\mathcal{K}_\ell \cap \partial\mathcal{K}_{\ell+1}, \\ \varepsilon_\ell^{ji}(\mathbf{0})\frac{\partial w_\ell^i}{\partial \mathbf{n}_\ell} + \varepsilon_{\ell+1}^{ji}(\mathbf{0})\frac{\partial w_{\ell+1}^i}{\partial \mathbf{n}_{\ell+1}} &= 0 \quad \text{on } \partial\mathcal{K}_\ell \cap \partial\mathcal{K}_{\ell+1}, \end{array} \right. \quad (\text{C.2})$$

$$\begin{aligned} F_m^j &= -u_m^j \Delta \eta - 2\varepsilon_m^{ji}(\mathbf{0}) \frac{\partial u_m^i}{\partial x_k} \frac{\partial \eta}{\partial x_k} + f_m^j \eta, \\ G_m^j &= g_m^j \eta - \alpha_m^j u_m^j \eta. \end{aligned}$$

The regularity of  $\mathbf{u}_m$  in a neighborhood of the point  $A$  is determined by the smoothness of  $\mathbf{w}_m$  near  $A$ . Using polar coordinates  $(r, \omega)$  in (C.2) we arrive at the system

$$\left\{ \begin{array}{l} -\varepsilon_m^{ji}(\mathbf{0}) \left( \frac{\partial^2 \bar{w}_m^i}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{w}_m^i}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{w}_m^i}{\partial \omega^2} \right) = \bar{F}_m^j \text{ in } \bar{S}_m, \quad m = \ell, \ell + 1, \\ \varepsilon_\ell^{ji}(\mathbf{0}) \frac{\partial \bar{w}_\ell^i}{\partial \omega}(r, 0) = \bar{G}_\ell^j(r, 0), \\ \varepsilon_{\ell+1}^{ji}(\mathbf{0}) \frac{\partial \bar{w}_{\ell+1}^i}{\partial \omega}(r, \omega_{\ell+1}) = \bar{G}_{\ell+1}^j(r, \omega_{\ell+1}), \\ \bar{w}_\ell^j(r, \omega_\ell) = \bar{w}_{\ell+1}^j(r, \omega_\ell), \\ \varepsilon_\ell^{ji}(\mathbf{0}) \frac{\partial \bar{w}_\ell^i}{\partial \omega}(r, \omega_\ell) = \varepsilon_{\ell+1}^{ji}(\mathbf{0}) \frac{\partial \bar{w}_{\ell+1}^i}{\partial \omega}(r, \omega_\ell), \end{array} \right. \quad (\text{C.3})$$

where  $\bar{S}_\ell$  and  $\bar{S}_{\ell+1}$ , respectively, is an infinite half-strip

$$\begin{aligned} \bar{S}_\ell &= \{(r, \omega) : r \in \mathbb{R}_+, 0 < \omega < \omega_\ell\}, \\ \bar{S}_{\ell+1} &= \{(r, \omega) : r \in \mathbb{R}_+, \omega_\ell < \omega < \omega_{\ell+1}\}, \end{aligned}$$

respectively, and  $\bar{\mathbf{w}}_m(r, \omega) = \mathbf{w}_m(x_1, x_2)$ ,  $\bar{\mathbf{F}}_m(r, \omega) = \mathbf{F}_m(x_1, x_2)$  and  $\bar{\mathbf{G}}_m(r, \omega) = \mathbf{G}_m(x_1, x_2)$ ,  $m = \ell, \ell + 1$ . Substituting  $r = e^\xi$ , we get the system of equations

$$\left\{ \begin{array}{l} -\varepsilon_m^{ji}(\mathbf{0}) \left( \frac{\partial^2 \tilde{w}_m^i}{\partial \xi^2} + \frac{\partial^2 \tilde{w}_m^i}{\partial \omega^2} \right) = \tilde{F}_m^j \text{ in } \tilde{S}_m, \quad m = \ell, \ell + 1, \\ \varepsilon_\ell^{ji}(\mathbf{0}) \frac{\partial \tilde{w}_\ell^i}{\partial \omega}(\xi, 0) = \tilde{G}_\ell^j(\xi, 0), \\ \varepsilon_{\ell+1}^{ji}(\mathbf{0}) \frac{\partial \tilde{w}_{\ell+1}^i}{\partial \omega}(\xi, \omega_{\ell+1}) = \tilde{G}_{\ell+1}^j(\xi, \omega_{\ell+1}), \\ \tilde{w}_\ell^j(\xi, \omega_\ell) = \tilde{w}_{\ell+1}^j(\xi, \omega_\ell), \\ \varepsilon_\ell^{ji}(\mathbf{0}) \frac{\partial \tilde{w}_\ell^i}{\partial \omega}(\xi, \omega_\ell) = \varepsilon_{\ell+1}^{ji}(\mathbf{0}) \frac{\partial \tilde{w}_{\ell+1}^i}{\partial \omega}(\xi, \omega_\ell), \end{array} \right. \quad (\text{C.4})$$

where  $\tilde{S}_\ell$  and  $\tilde{S}_{\ell+1}$ , respectively, denotes an infinite strip

$$\begin{aligned} \tilde{S}_\ell &= \{(\xi, \omega) : \xi \in \mathbb{R}, 0 < \omega < \omega_\ell\}, \\ \tilde{S}_{\ell+1} &= \{(\xi, \omega) : \xi \in \mathbb{R}, \omega_\ell < \omega < \omega_{\ell+1}\}, \end{aligned}$$

respectively, and  $\tilde{\mathbf{w}}(\xi, \omega) = \mathbf{w}(x_1, x_2)$ ,  $\tilde{\mathbf{F}}_m(\xi, \omega)e^{-2\xi} = \bar{\mathbf{F}}_m(r, \omega)$ ,  $\tilde{\mathbf{G}}_m(\xi, \omega)e^{-\xi} = \bar{\mathbf{G}}_m(r, \omega)$ . We apply the complex Fourier transform  $\mathcal{F}_{\xi \rightarrow \lambda}$  with respect to real variable  $\xi \in \mathbb{R}$ ,

$$[\mathcal{F}_{\xi \rightarrow \lambda} \phi(\xi)](\lambda) = \hat{\phi}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(\xi) e^{-i\lambda\xi} d\xi, \quad \lambda \in \mathbb{C},$$

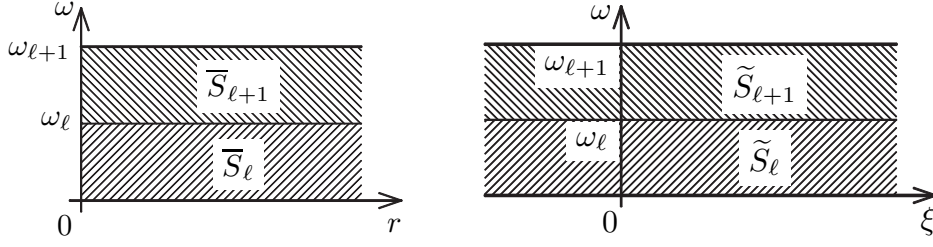


Figure C.3: The infinite half-strip  $\overline{S} = \overline{S}_\ell \cup \overline{S}_{\ell+1}$  and the infinite strip  $\tilde{S} = \tilde{S}_\ell \cup \tilde{S}_{\ell+1}$ .

to transform (C.4) to the one-dimensional problem

$$\left\{ \begin{array}{l} -\varepsilon_\ell^{ji}(\mathbf{0}) \left( (i\lambda)^2 \widehat{w}_\ell^i + \frac{\partial^2 \widehat{w}_\ell^i}{\partial \omega^2} \right) = \widehat{F}_\ell^j \text{ for } \omega \in (0, \omega_\ell), \\ -\varepsilon_{\ell+1}^{ji}(\mathbf{0}) \left( (i\lambda)^2 \widehat{w}_{\ell+1}^i + \frac{\partial^2 \widehat{w}_{\ell+1}^i}{\partial \omega^2} \right) = \widehat{F}_{\ell+1}^j \text{ for } \omega \in (\omega_\ell, \omega_{\ell+1}), \\ \varepsilon_\ell^{ji}(\mathbf{0}) \frac{\partial \widehat{w}_\ell^i}{\partial \omega}(\lambda, 0) = \widehat{G}_\ell^j(\lambda, 0), \\ \varepsilon_{\ell+1}^{ji}(\mathbf{0}) \frac{\partial \widehat{w}_{\ell+1}^i}{\partial \omega}(\lambda, \omega_{\ell+1}) = \widehat{G}_{\ell+1}^j(\lambda, \omega_{\ell+1}), \\ \widehat{w}_\ell^j(\lambda, \omega_\ell) = \widehat{w}_{\ell+1}^j(\lambda, \omega_\ell), \\ \varepsilon_\ell^{ji}(\mathbf{0}) \frac{\partial \widehat{w}_\ell^i}{\partial \omega}(\lambda, \omega_\ell) = \varepsilon_{\ell+1}^{ji}(\mathbf{0}) \frac{\partial \widehat{w}_{\ell+1}^i}{\partial \omega}(\lambda, \omega_\ell) \end{array} \right. \quad (\text{C.5})$$

with complex parameter  $\lambda$ . Solvability of the parameter dependent boundary value problems were studied in [17]. Roughly speaking, the operator pencil  $\widehat{\mathfrak{A}}_A(\lambda)$ , associated with the parameter dependent boundary value problem (C.5), is an isomorphism for all complex parameters  $\lambda \in \mathbb{C}$  except at certain isolated points – the eigenvalues of  $\widehat{\mathfrak{A}}_A(\lambda)$  (for precise definition of eigenvalues and corresponding eigensolutions we refer to monograph [17]). As well-known, the regularity of the weak solution  $\mathbf{u}_\ell \in \mathbf{W}_\ell^{1,2}$  (the existence of the strong solution, i.e. whether or not  $\mathbf{u}_\ell \in \mathbf{W}_\ell^{2,2}$ ) depends on the distribution of the eigenvalues  $\lambda$  of the operator pencil  $\widehat{\mathfrak{A}}_A(\lambda)$  in the strip  $\text{Im } \lambda \in (-1, 0)$ . This assertion is expressed by the following theorem, which is a classical result, see [6], [13], [14], [15], [17]:

**Theorem Appendix C.1** (Regularity theorem in an infinite angle). *Let  $\mathbf{w}_\ell \in \mathbf{W}_\ell^{1,2}(\mathcal{K}_\ell)$  be the uniquely determined weak solution of (C.2),  $\mathbf{F}_\ell \in \mathbf{L}_\ell^2(\mathcal{K}_\ell)$ ,  $\mathbf{G}_\ell \in \mathbf{W}_{\ell,\Gamma}^{1/2,2}(\mathcal{K}_\ell)$ . If the strip  $\text{Im } \lambda \in (-1, 0)$  is free of eigenvalues of the operator pencil  $\widehat{\mathfrak{A}}_A(\lambda)$ , then  $\mathbf{w}_\ell \in \mathbf{W}_\ell^{2,2}(\mathcal{K}_\ell)$  and*

$$\|\mathbf{w}_\ell\|_{\mathbf{W}_\ell^{2,2}(\mathcal{K}_\ell)} \leq c \left( \|\mathbf{F}_\ell\|_{\mathbf{L}_\ell^2(\mathcal{K}_\ell)} + \|\mathbf{G}_\ell\|_{\mathbf{W}_{\ell,\Gamma}^{1/2,2}(\mathcal{K}_\ell)} \right).$$

*Proof.* Theorem Appendix C.1 is a consequence of [17, §7, Theorem 7.5] and the expansion of the solution [17, §7, (7.10)].  $\square$

The characteristic determinants and the distribution of the eigenvalues of  $\widehat{\mathfrak{A}}_A(\lambda)$

Every  $\lambda_0 \in \mathbb{C}$  such that  $\ker \widehat{\mathfrak{A}}_A(\lambda_0) \neq \{\mathbf{0}\}$  is said to be an eigenvalue of  $\widehat{\mathfrak{A}}_A(\lambda)$ . The distribution of the eigenvalues of the operator  $\widehat{\mathfrak{A}}_A(\lambda)$  plays crucial role in the regularity results of the solution, see Theorem Appendix C.1. In order to calculate the eigenvalues of the operator pencil  $\widehat{\mathfrak{A}}_A(\lambda)$  we look those  $\lambda$ , for which there exists the nontrivial solution of the system (C.5) with the vanishing right-hand side. The general solution  $[\widehat{e}_m^1, \widehat{e}_m^2]$  of the homogeneous equations

$$\begin{aligned} -\varepsilon_m^{11}(\mathbf{0}) \left( (i\lambda)^2 \widehat{e}_m^1 + \frac{\partial^2 \widehat{e}_m^1}{\partial \omega^2} \right) - \varepsilon_m^{12}(\mathbf{0}) \left( (i\lambda)^2 \widehat{e}_m^2 + \frac{\partial^2 \widehat{e}_m^2}{\partial \omega^2} \right) &= 0, \\ -\varepsilon_m^{21}(\mathbf{0}) \left( (i\lambda)^2 \widehat{e}_m^1 + \frac{\partial^2 \widehat{e}_m^1}{\partial \omega^2} \right) - \varepsilon_m^{22}(\mathbf{0}) \left( (i\lambda)^2 \widehat{e}_m^2 + \frac{\partial^2 \widehat{e}_m^2}{\partial \omega^2} \right) &= 0, \end{aligned}$$

$m = \ell, \ell + 1$ , has the form (recall that  $\varepsilon_m^{ji}$  is a positive definite matrix)

$$\begin{cases} \widehat{e}_\ell^1 &= C_1 \cos(i\lambda\omega) + C_2 \sin(i\lambda\omega) \text{ for } \omega \in (0, \omega_\ell), \\ \widehat{e}_\ell^2 &= C_3 \cos(i\lambda\omega) + C_4 \sin(i\lambda\omega) \text{ for } \omega \in (0, \omega_\ell), \\ \widehat{e}_{\ell+1}^1 &= C_5 \cos(i\lambda\omega) + C_6 \sin(i\lambda\omega) \text{ for } \omega \in (\omega_\ell, \omega_{\ell+1}), \\ \widehat{e}_{\ell+1}^2 &= C_7 \cos(i\lambda\omega) + C_8 \sin(i\lambda\omega) \text{ for } \omega \in (\omega_\ell, \omega_{\ell+1}). \end{cases} \quad (\text{C.6})$$

The eigenvalues of  $\widehat{\mathfrak{A}}_A(\lambda)$  are zeros of the determinant  $D_A(\lambda)$  of corresponding matrix of coefficients  $C_1, \dots, C_8$  (substituting the general solution (C.6) to the corresponding boundary conditions and transmission conditions, respectively, we get the homogeneous linear system of eight equations with unknowns  $C_1, \dots, C_8$ ). Computation of  $D_A(\lambda)$  leads to the transcendent equation

$$D_A(\lambda) = D_A^{11}(\lambda)D_A^{22}(\lambda) - D_A^{12}(\lambda)D_A^{21}(\lambda) = 0, \quad (\text{C.7})$$

where

$$\begin{aligned} D_A^{11}(\lambda) &= \varepsilon_\ell^{11}(\mathbf{0}) \sin(i\lambda\omega_\ell) \cos[i\lambda(\omega_{\ell+1} - \omega_\ell)] \\ &\quad + \varepsilon_{\ell+1}^{11}(\mathbf{0}) \cos(i\lambda\omega_\ell) \sin[i\lambda(\omega_{\ell+1} - \omega_\ell)], \\ D_A^{12}(\lambda) &= \varepsilon_\ell^{12}(\mathbf{0}) \sin(i\lambda\omega_\ell) \cos[i\lambda(\omega_{\ell+1} - \omega_\ell)] \\ &\quad + \varepsilon_{\ell+1}^{12}(\mathbf{0}) \cos(i\lambda\omega_\ell) \sin[i\lambda(\omega_{\ell+1} - \omega_\ell)], \\ D_A^{21}(\lambda) &= \varepsilon_\ell^{21}(\mathbf{0}) \sin(i\lambda\omega_\ell) \cos[i\lambda(\omega_{\ell+1} - \omega_\ell)] \\ &\quad + \varepsilon_{\ell+1}^{21}(\mathbf{0}) \cos(i\lambda\omega_\ell) \sin[i\lambda(\omega_{\ell+1} - \omega_\ell)], \\ D_A^{22}(\lambda) &= \varepsilon_\ell^{22}(\mathbf{0}) \sin(i\lambda\omega_\ell) \cos[i\lambda(\omega_{\ell+1} - \omega_\ell)] \\ &\quad + \varepsilon_{\ell+1}^{22}(\mathbf{0}) \cos(i\lambda\omega_\ell) \sin[i\lambda(\omega_{\ell+1} - \omega_\ell)]. \end{aligned}$$

The roots of the equation  $D_A(\lambda) = 0$  are the eigenvalues of  $\widehat{\mathfrak{A}}_A(\lambda)$ .

Taking into account the special type of geometry, namely  $\omega_\ell = \omega_{\ell+1}/2$ , (C.7) simplifies into

$$\begin{aligned} \frac{1}{2} [(\varepsilon_\ell^{11}(\mathbf{0}) + \varepsilon_{\ell+1}^{11}(\mathbf{0}))(\varepsilon_\ell^{22}(\mathbf{0}) + \varepsilon_{\ell+1}^{22}(\mathbf{0})) \\ - (\varepsilon_\ell^{12}(\mathbf{0}) + \varepsilon_{\ell+1}^{12}(\mathbf{0}))(\varepsilon_\ell^{21}(\mathbf{0}) + \varepsilon_{\ell+1}^{21}(\mathbf{0}))] \sin(2i\lambda\omega_\ell) = 0. \end{aligned} \quad (\text{C.8})$$

Since both matrices,  $\varepsilon_\ell^{ij}$  and  $\varepsilon_{\ell+1}^{ij}$ , are considered to be positive definite, we get

$$\sin(2i\lambda\omega_\ell) = 0, \quad (\text{C.9})$$

from whence we obtain

$$i\lambda = \frac{k\pi}{2\omega_\ell}, \quad k \in \mathbb{Z}.$$

Now it is clear that for  $\omega_\ell \in (0, \pi/2]$  there are no roots of the equation  $D_A(\lambda) = 0$  such that  $\text{Im } \lambda \in (-1, 0)$ .

**Corollary 1.** Let  $\mathbf{u}_\ell \in \mathbf{W}_\ell^{1,2}$  be the uniquely determined weak solution of (C.1),  $\mathbf{f}_\ell \in \mathbf{L}_\ell^2$ ,  $\mathbf{g}_\ell \in \mathbf{W}_{\ell,\Gamma}^{1/2,2}$ . Since the strip  $\text{Im } \lambda \in (-1, 0)$  is free of eigenvalues of the operator  $\widehat{\mathfrak{A}}_A(\lambda)$ , we have  $\mathbf{u}_\ell \in \mathbf{W}_\ell^{2,2}$  and

$$\|\mathbf{u}_\ell\|_{\mathbf{W}_\ell^{2,2}} \leq c \left( \|\mathbf{f}_\ell\|_{\mathbf{L}_\ell^2} + \|\mathbf{g}_\ell\|_{\mathbf{W}_{\ell,\Gamma}^{1/2,2}} \right).$$

*Sketch of the proof.* The assertion follows from Theorem Appendix C.1 and the determinant equation (C.7). (C.7) implies that for  $\omega_{\ell+1} \leq \pi$ ,  $\omega_\ell = \omega_{\ell+1}/2$  there are no eigenvalues of the operator  $\widehat{\mathfrak{A}}_A(\lambda)$  in the strip  $\text{Im } \lambda \in (-1, 0)$ . Hence  $\mathbf{u}_\ell \in \mathbf{W}_\ell^{2,2}(\mathcal{K}_\ell)$ . Now consider the weak solution  $\mathbf{u}_\ell \in \mathbf{W}_\ell^{1,2}$  of (C.1). Let  $\mathcal{M}_\ell$  be the set of all boundary corner points  $A \in \partial\Omega_\ell$ ,  $\ell = 1, \dots, M$ . We have

$$\mathbf{u}_\ell = \left( 1 - \sum_{A \in \mathcal{M}_\ell} \eta_A \right) \mathbf{u}_\ell + \sum_{A \in \mathcal{M}_\ell} \eta_A \mathbf{u}_\ell \quad (\text{C.10})$$

$$= \left( 1 - \sum_{A \in \mathcal{M}_\ell} \eta_A \right) \mathbf{u}_\ell + \sum_{A \in \mathcal{M}_\ell} \mathbf{w}_\ell. \quad (\text{C.11})$$

The regularity of the first term on the right-hand side follows from the interior regularity. The smoothness of the second term follows from the regularity result in an infinite angle  $\mathcal{K}_\ell$ , Theorem Appendix C.1.  $\square$

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